McKay correspondence and derived equivalences

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Introduction (Français)

L'objectif de cette thèse est d'étudier certains aspects de la correspondance de McKay dans une situation provenant de la théorie des représentations modulaires des groupes finis.

Sur la correspondance de McKay

L'étude de la correspondance de McKay débute dans les années '80 (travaux de McKay, Gonzalez-Sprinberg et Verdier) avec la description d'une relation entre la cohomologie de la résolution minimale X d'une variété $\mathbb{C}^2/G$ et l'ensemble des représentations irréductibles de G, où G est un sous-groupe fini de $\mathrm{SL}_2(\mathbb{C})$.

Plus précisément, soit G l'un des groupes du tableau suivant:

Type $\mathcal{T}$	Groupe G	Nom
A_n	μ_{n+1}	cyclique d'ordre n
D_n	BD_{4n}	binaire diédral
E_6	$\mathbb{T}$	binaire thetrahédral
E_7	$\mathbb{O}$	binaire octohédral
E_8	$\mathbb{I}$	binaire icosahédral

Table 1: Liste des sous-groupes finis de $\mathrm{SL}_2(\mathbb{C})$.

Les composantes irréductibles du lieu exceptionnel sont des courbes E_i, $i \in I$, où I est un ensemble fini. Chacune de ces courbes est isomorphe à l'espace projectif $\mathbb{P}^1$ et est d'auto-intersection -2. De plus, l'égalité suivante a lieu:

$$H_2 := H_2(X, \mathbb{Z}) = \bigoplus_{i \in I} \mathbb{Z}[E_i].$$

La forme bilinéaire

$$\begin{array}{rcc} (,)_{\mathrm{sing}} : H_2 \times H_2 & \to & \mathbb{Z} \\ (E_i, E_j) & \mapsto & \left\{ \begin{array}{c} (E_i, E_j)_{\mathrm{sing}} = 0 \text{ ou } 1, \forall i, j \in I \\ (E_i, E_i)_{\mathrm{sing}} = -2, \forall i \in I \end{array} \right. \end{array}$$

permet de définir un graphe $\mathcal{G}$ dont les sommets sont les courbes E_i, deux sommets E_i et E_j étant reliés par une arête si et seulement si $(E_i, E_j)_{\mathrm{sing}}$ vaut 1.

D'autre part, soit $\mathrm{Irr}(G)$ l'ensemble des représentations irréductibles de G et soit ρ_{nat} la représentation de dimension deux correspondante à l'inclusion $G \hookrightarrow \mathrm{SL}_2(\mathbb{C})$. Étant donné $\rho \in \mathrm{Irr}(G)$, considérons la représentation $\rho \otimes \rho_{\mathrm{nat}}$ que l'on décompose en une somme directe de représentations irréductibles. Celles-ci peuvent alors être considérées comme les sommets d'un graphe $\widetilde{\mathcal{G}}$, deux sommets ρ_i et ρ_j étant reliés par une arête si et seulement si ρ_j apparaît dans la décomposition de $\rho_i \otimes \rho_{\mathrm{nat}}$.

La correspondance de McKay affirme que si le groupe G est du type $\mathcal{T}$ (comme dans la Table 1), alors le graphe $\mathcal{G}$ est le diagramme de Dynkin de type $\mathcal{T}$ et le graphe $\widetilde{\mathcal{G}}$ est le diagramme de Dynkin élargi.

Dans le cas d'un groupe de Klein, Gonzalez-Sprinberg et Verdier ont montré en 1983 l'existence de faisceaux $\{F_\rho\}_{\rho \in \mathrm{Irr}(G)}$ sur X, dont la première classe de Chern forme une base de la cohomologie de X.

Deux années plus tard, la correspondance de McKay franchit les barrières mathématiques. Les physiciens Dixon, Harvey, Vafa et Witten, dans le cadre de la théorie des cordes, sont les premiers à poser la question suivante: si G est un sous-groupe fini de $\mathrm{SL}_3(\mathbb{C})$ tel que le quotient $\mathbb{C}^3/G$ admet une résolution crépante X, quel lien existe-t-il entre le nombre d'Euler $e(X)$ et le nombre de représentations irréductibles, voir le nombre de classes de conjugaison, de G ?

Dans les année '90, Ito et Nakamura apportent de nombreuses contributions au développement de la correspondance de McKay. Ils montrent en effet que, dans le cas des groupes de Klein, la résolution de Du Val n'est autre qu'une certaine composante $G-$invariante du schéma de Hilbert de $\#G$ points sur $\mathbb{C}^2$, appelée $G-$schéma de Hilbet de $\mathbb{C}^2$ et notée $G - \mathrm{Hilb}\mathbb{A}^2$. La notion de $G-$schéma de Hilbert peut plus généralement être considérée pour une variété projective lisse Y, de dimension $n \geq 2$, et G un sous-groupe fini d'automorphismes de Y. Le schéma $G - \mathrm{Hilb}Y$ s'avère être un bon candidat pour une résolution des singularités de Y/G.

Une dernière remarque peut être faite concernant les diviseurs canoniques de $\mathbb{C}^2/G$ et sa résolution minimale X. Si $f : X \to \mathbb{C}^2/G$ est le morphisme de résolution, et si l'on désigne respectivement par K_X et $K_{\mathbb{C}^2/G}$ les diviseurs canoniques de X et $\mathbb{C}^2/G$, alors l'égalité suivante a lieu:

$$K_X = f^*(K_{\mathbb{C}^2/G}).$$

Reid et al font référence à cette égalité sous le nom de propriété de crépance du morphisme de résolution, ceci du au fait que la discrépance de f est nulle.

Pour conclure, qu'est ce que la correspondance de McKay ? Dans sa conférence au séminaire Bourbaki, en 1999, Reid propose la formulation suivante:

Soit Y une variété lisse et G un groupe d'automorphismes de X. Soit $f : X \to Y/G$ une résolution de singularités. Alors, la réponse à toute question sur la géométrie de X est la géométrie $G-$équivariante de Y.

La problématique

Les résultats précédents ont donné lieu à des questions variées apparaissant naturellement dans l'étude de la correspondance de McKay. En voici une liste non-exhaustive.

1. Si G est un sous-groupe fini de $\mathrm{SL}_n(\mathbb{C})$ agissant fidèlement sur $\mathbb{A}^n$, l'espace affine de dimension n, le $G-$schéma de Hilbert est-il une variété lisse ? Si oui, est-il de plus une résolution crépante de la singularité quotient $\mathbb{A}^n/G$?

2. Plus généralement, soit Y une variété quasi-projective lisse et G un sous-groupe fini du groupe des automorphismes de Y tel que $K_{Y/G} = 0$. Le quotient Y/G possède-t-il une résolution crépante ? Dans l'affirmative, $G - \mathrm{Hilb}Y$, à supposer qu'il soit lisse, en est-il une ?

3. Dans la situation précédente, comment les propriétés algébrique du groupe G interviennent-elles dans la description de $G - \mathrm{Hilb}Y$? Dans le cas d'une résolution crépante X, est-il vrai que le nombre d'Euler $e_G(X)$ est le nombre des classes de conjugaison de G ?

4. Avec les mêmes notations que dans (2), supposons que $f : G{-}\mathrm{Hilb}Y \to Y$ est une résolution crépante. Existe-t-il une équivalence des catégories dérivées
$$F : D^b(G - \mathrm{Hilb}Y) \xrightarrow{\sim} D^b_G(Y) \ ?$$
Si une telle équivalence a lieu, le foncteur F est-il une transformée de Fourier-Mukai ?

De nombreux travaux ont paru ces dernières années pour tenter de répondre à ces questions. Entre 1995 et 2000, Dais, Ziegler et al, montrent que les singularités quotients qui peuvent être décrites localement comme des intersections complètes admettent des résolutions projectives crépantes. En 2002, Bezrukavnikov et Kaledin montrent que dans le cas symplectique la propriété 4 reste vraie, c'est a dire la correspondance de McKay a lieu. Leur résultat implique en particulier que, dans le cas sympectique, toute résolution crépante est un espace de modules (voir la notion de $G-$constellation), mais ce n'est pas forcement le $G-$schéma de Hilbert.

Le cas le plus étudié reste celui d'un sous-groupe fini $G \subset \mathrm{SL}_n(\mathbb{C})$. L'intégration motivique à la Denef-Loeser s'avère être un outil puissant permettant de donner une réponse positive à la question posée par Vafa et al. Elle ne répond cependant pas au besoin de décrire une résolution crépante de $\mathbb{A}^n/G$, et ne permet en particulier pas de décrire le $G-$schéma de Hilbert de $\mathbb{A}^n$.

En dimension $n \leq 3$, le $G-$schéma de Hilbert fournit une résolution crépante du quotient $\mathbb{A}^n/G$ (ceci ne se produit cependant que très rarement en dimension supérieure). Pour $n = 3$, les techniques utilisés pour la description de $G - \mathrm{Hilb}\mathbb{A}^3$ se réduisent à une analyse au cas par cas selon la caractéristique du groupe G. Dans le cas d'un groupe abélien, l'étude repose sur les propriétés des variétés toriques (cf. Craw, Ito, Nakamura, Reid). Très peu de travaux sont en revanche disponibles dans le cas d'un groupe non-commutatif (Markushevich pour H_{168} et plus récemment Leng pour des groupes trihédraux et Térouanne pour des sous-groupes des groupes de Weyl).

En 2001, Bridgeland, King et Reid montrent que la correspondance de McKay dérivée (4) a lieu dans les conditions suivantes. Soit Y une variété

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "\pi"] \\ & Y/G & \end{tikzcd}$$

Figure 1: Le diagramme BKR.

lisse et G un groupe d'automorphismes de Y. Notons X la composante irréductible de $G - \mathrm{Hilb}Y$ qui contient les orbites libres de G. On suppose que la dimension du produit fibré est telle que:

$$X \times_{Y/G} X \leq \dim(Y). \tag{0.0.1}$$

La transformé de Fourier-Mukai peut alors être utilisée comme un quantificateur des caractéristiques géométriques de X en lieu et place de la $K-$théorie ou de la cohomologie. Plus précisément, si Z désigne le schéma universel fermé de $X \times Y$ et si l'on dénote respectivement par p et q les projections de Z sur X et Y (cf. Figure 1), alors le foncteur $Rq_* \circ Lp^*$ fourni une équivalence de catégories dérivées:

$$Rq_* \circ Lp^* : D^b(X) \to D^b_G(Y).$$

Les résultats

Face à une problématique si variée et à l'abondance de résultats dans la correspondance de McKay, se pose la question de ce qu'il reste à faire. Heureusement plein de choses !

Hormis le cas intersection complète, nous connaissons très peu de classes de groupes $G \subset \mathrm{SL}_n(\mathbb{C})$ tels que, pour tout n, le $G-$schéma de Hilbert de $\mathbb{A}^n$ est à la fois lisse et fournit une résolution crépante du quotient $\mathbb{A}^n/G$.

Nous démontrons dans la première partie de cette thèse que pour la classe $\{G_n := \mu_{2^n-1}\}_{n\geq 2}$ des groupes cycliques d'ordre $2^n - 1$, agissant par les poids $2^{i-1}, i \in \{1, \ldots, n\}$ sur l'espace affine $\mathbb{A}^n$, le G_n-schéma de Hilbert est une variété lisse et également une résolution crépante des singularités de Gorenstein de $\mathbb{A}^n/G_n$. Le résultat est le suivant:

Théorème 1.1 *Pour tout entier positif n, le $\mu_{2^n-1}-$schéma de Hilbert de $\mathbb{A}^n$ est une résolution crépante des singularités quotients $\mathbb{A}^n/\mu_{2^n-1}$.*

Remarquons que dans le cas envisagé dans ce théorème, les techniques du cas symplectique ou intersection complète ne peuvent être appliquées. La condition de Bridgeland-King-Reod (BKR 0.0.1) n'est pas vérifiée non plus. En gros, l'idée est que le lieu exceptionnel est trop grand (il contient des diviseurs). Ainsi l'inégalité $2 \times \dim\ (fibre) \leq n+1$ n'a pas lieu. La démonstration du Théorème 1.1 utilise en fait les propriétés des variétés toriques. Il s'agit d'une technique de déformation basée sur la notion de $G-$graphe de Nakamura. La listé découle du dénombrements des cônes.

D'après la discussion précédente, on s'attend à ce que la catégorie dérivée bornées des faisceaux cohérents sur le G_n-schéma de Hilbert soit équivalente à la catégorie dérivée bornées des faisceaux cohérents G_n-équivariants de $\mathbb{A}^n$. Ce résultat est démontré dans la deuxième partie de cette thèse. La démonstration consiste en la factorisation du morphisme de Hilbert-Chow

$$f : \mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n \to \mathbb{A}^n/\mu_{2^n-1}$$

en une suite de résolutions partielles. Chacune de ces résolutions est une contractions divisorielle et le résultat découle des équivalences entre catégories dérivées des champs de Deligne-Mumford lisses associés au résolutions partielles. Le deuxième chapitre contient des rappels sur les champs algébriques dont les preuves de certaines propriétés dont la démonstration n'est pas toujours très détaillée dans la littérature. En particulier, nous construisons explicitement un champs de Deligne-Mumford lisse.

Le troisième chapitre consiste en la description explicite de la décomposition du morphisme de Hilbert-Chow en vu de la démonstration du théorème:

Théorème 3.1 *La catégorie dérivée bornée des faisceaux cohérents $\mu_{2^n-1}-$équivariants sur $\mathbb{A}^n$ est équivalente à la catégorie dérivée bornée des faisceaux cohérents de $\mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n$.*

Ces résultats sont valables en toute caractéristique p première à l'ordre 2^n-1 du groupe G_n. La classe des groupes $G_n, n \geq 2$, provient d'un corps fini κ de caractéristique 2, en considérant l'action par multiplication de κ^* sur κ^n, vu comme espace vectoriel sur $\mathbb{F}_2$. Dans la dernière partie de cette thèse, nous concluons sur une équivalence entre la catégorie dérivée des modules gradués du bloc principal de $\kappa[\mathrm{SL}_2(\kappa)]$ et la catégorie des faisceaux κ^*-équivariants sur le G_n-schéma de Hilbert. Cela donne une réalisation géométrique des représentations modulaires de $\mathrm{SL}_2(\kappa)$, via la dualité de Koszul, vers la correspondance de McKay.

Pour finir, l'Annexe A contient des travaux en cours sur la description algorithmique du $G-$schéma de Hilbert de $\mathbb{A}^3$ dans le cas où G est un groupe trihédral. Ces sont les premiers pas vers une possible description explicite des $G-$schémas de Hilbert dans le cas des groupes non-commutatifs.

Dans la suite, on marque par $\blacksquare$ la fin d'une démonstration et par $\clubsuit$ la fin d'une remarque ou d'une notation.

Chapter 1

Crepancy for toric varieties

Introduction

This chapter generalizes to higher dimensions some previously-known algorithms of resolving toric quotient singularities in order to obtain crepant resolutions.

Let n be a non-negative integer. We denote by $G_n := \mu_{2^n-1} \subset \mathbb{C}^*$ the cyclic group of order 2^n-1 generated by ε, a primitive root of unity of order 2^n-1. Let this group act by weights $1, 2, 2^2, \ldots, 2^{n-1}$ on the affine space $\mathbb{A}^n$. This is the same as the action of the subgroup H_n of $SL_n(\mathbb{C})$, generated by the diagonal matrix $g_n := \mathrm{diag}(\varepsilon, \varepsilon^2, \varepsilon^{2^2}, \ldots, \varepsilon^{2^{n-1}})$, by multiplication on $\mathbb{A}^n$. The quotient $\mathbb{A}^n/G_n$ – which is the same as $\mathbb{A}^n/H_n$ – has one isolated singularity, at the origin. In the sequel, we prove the following theorem:

Theorem 1.1. *For any positive integer n, the $\mu_{2^n-1}-$Hilbert scheme of $\mathbb{A}^n$ is a crepant resolution of singularities of the quotient $\mathbb{A}^n/\mu_{2^n-1}$.*

The first question that one should ask is **where** does this group come from? The answer is given in Chapter 3 and is closely related to Broué's conjecture for modular representations of finite groups, as stated there.

Next, one should see **what** are the notions introduced in the previous theorem. For crepant resolution of singularities and their properties, see Section 1.1, part 1.1.2. Here, we have Calabi-Yau varieties, so crepancy means that the canonical sheaf $\omega_{\mu_{2^n-1}-\mathrm{Hilb}\mathbb{C}^n}$ and the structure sheaf $\mathcal{O}_{\mu_{2^n-1}-\mathrm{Hilb}\mathbb{C}^n}$ are isomorphic. For a finite group G, we recall in 1.1.3 the definition of the $G-$Hilbert scheme. Part 1.2 introduces the notion of $G-$graph and gives the link between $G-$Hilbert schemes and $G-$graphs as the main tools for the proof.

Finally, one should ask **why** is this example an interesting one? The quotient $\mathbb{A}^n/\mu_{2^n-1}$ is a Gorenstein canonical singularity and has only one isolated singularity at the origin. In order to resolve the singularities, we will provide a simplicial decomposition of its defining fan σ_0 into sub-cones such that the resulting variety is $\mu_{2^n-1}-\mathrm{Hilb}\mathbb{C}^n$. We prove that $\mu_{2^n-1}-\mathrm{Hilb}\mathbb{C}^n$ is smooth and it has the crepancy property. As far as the author knows, for large n, this is the only known example of cyclic groups acting on $n-$dimensional affine space (without non-zero fixed subspace) such that the quotient admits a crepant toric resolution. We remark that the methods of [7] can't be applied in this case — the condition on the fiber product of Theorem 1.1 of the cited paper is not satisfied. Using Definition 5.2 of [12] and Watanabe's Theorem (see Theorem 5.3 of the same paper), we see that the group μ_{2^n-1} doesn't give rise to a complete intersection singularity. In particular, the techniques of [11] are not applicable. We are not in the symplectic case, so [5] doesn't apply either.

1.1 Generalities

In the first part of this section, we recall some basic results on toric varieties as in [15], [25] or [34], especially on how to compute Cartier and Weil divisors. Subsection 1.1.2 contains the definition of a crepant resolution and some of its properties, following [37].

Before starting, we recall some classical notions.

We denote by $\langle , \rangle$ the scalar product in $\mathbb{R}^n$.

For a positive integer we denote by $\pmod n$ the remainder of the division by n. For an integer i, the notation $i \pmod n$ stands for the unique non-negative integer j between 0 and $n-1$ such that $i-j$ is a multiple of n.

Let X be a quasi-projective, smooth variety over $\mathbb{C}$.

Definition 1.2. *1. ([36], Section 5, "cluster") A cluster in X is a zero-dimensional sub-scheme Z.*

2. ([7], Section 1, "$G-$cluster") Let G be a finite group acting on X. A $G-$cluster is a $G-$invariant cluster such that the global sections $\Gamma(Z, \mathcal{O}_Z)$ are isomorphic to the regular representation $\mathbb{C}[G]$ of G.

Remark 1.3. A $G-$cluster Z has length $\#G$ – the cardinal of G. Moreover, $\mathcal{O}_Z := \mathcal{O}_X/\mathcal{I}_Z$ is a finite dimensional $\mathbb{C}-$vector space, where $\mathcal{I}_Z$ is the ideal defining the $G-$cluster Z. Any free $G-$orbit is a $G-$cluster. ♣

1.1.1 Toric varieties

Let n be a non-negative integer. In the sequel, we denote a lattice by $L, N \ldots$ The notation $N_{\mathbb{R}}$ stands for $N \otimes_{\mathbb{Z}} \mathbb{R}$, this is the vector space generated by the lattice N; we also denote it by V or $\langle N \rangle$. We call N_0 the lattice $\mathbb{Z}^n$. We denote by $\{e_i\}_{1 \le i \le n}$ the canonical basis of $\mathbb{Z}^n$. With the notations above, let $\{v_1, \ldots, v_t\}$ be a finite set of vectors in V, the vector space associated to a lattice N. The set

$$\{\sum_{i=1}^{t} a_i v_i | a_i \geq 0, \forall 1 \leq i \leq t\}$$

is called the convex polyhedral cone associated to $v_1, \ldots, v_t$. We denote it by $\langle v_1, \ldots, v_t \rangle$ and call the vectors $v_1, \ldots, v_t$ its generators. A convex polyhedral cone $\langle v_1, \ldots, v_t \rangle$ of a lattice N is:

• a simplex (or simplicial) if it can be generated by a subset of $\{v_1, \ldots, v_t\}$ made of linearly independent vectors;

• rational if all the vectors v_i are in N;

• strongly convex if it contains no nonzero linear subspace (see [15], Section 1.2, (13) for equivalent definitions).

In the sequel, a strongly convex rational polyhedral cone is called a cone and is denoted by $\sigma, \tau \ldots$ If σ and τ are two cones such that σ is contained in τ, we say that σ is a sub-cone of τ and we write then $\sigma \prec \tau$. We call σ_0 the cone generated in N_0 by $\{e_i\}_{1 \leq i \leq n}$. A cone generated by one vector is called a ray and will be denoted ρ. The dimension of a cone is the dimension of the vector space it generates. For a cone σ, let $\sigma(1)$ be the set of all sub-cones of dimension one. A face of a cone τ is a cone $\sigma \prec \tau$, with $\dim \sigma = \dim \tau - 1$. A fan is a collection of cones, denoted in general by Δ, such that a face of each cone of Δ is a cone in Δ and the intersection of two cones is a face of each. For a fan Δ, the set of all cones of dimension k is denoted by $\Delta(k)$. We say that a fan is simplicial if all its cones are.

For a lattice N we denote by $N^\vee$ the lattice $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and call it the dual of N. We denote by $\{f_i\}_{1 \leq i \leq n}$ the dual basis of $\{e_i\}_{1 \leq i \leq n}$ and we put M_0 to be the additive semi-group generated by 0 and $\{f_i\}_{1 \leq i \leq n}$. For a cone σ, its dual is the set $\sigma^\vee := \{u \in N^\vee_{\mathbb{R}} | \langle v, u \rangle \geq 0, \forall v \in \sigma\}$. We denote by $\sigma^\perp$ the set $\{u \in N^\vee_{\mathbb{R}} | \langle v, u \rangle = 0, \forall v \in \sigma\}$. Both $\sigma^\vee$ and $\sigma^\perp$ are cones in the lattice $N^\vee$. We put M^σ the set $\sigma^\perp \cap N^\vee$. The dual of a cone σ determines a commutative semi-group $S_\sigma := \sigma^\vee \cap N^\vee = \{u \in N^\vee | \langle v, u \rangle \geq 0, \forall v \in \sigma\}$. This defines an affine variety $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$. For a fan Δ, and σ and τ two cones in it, $U_{\sigma \cap \tau}$ is an open subset in U_σ and in U_τ. So we can glue U_σ and U_τ along $U_{\sigma \cap \tau}$. Thus, any fan Δ determines a variety, denoted by $X(\Delta)$ and called the toric variety associated to Δ.

We follow [15], Chapter 3 to recall how to describe divisors on toric varieties. We fix a lattice N and a fan Δ. As usual, $N^\vee$ is the dual lattice for N. For a cone σ, we denote by N_σ the sublattice of N generated as a group by $\sigma \cap N$ and we put $N(\sigma) = N/N_\sigma$. For $\sigma \prec \tau$ cones of Δ, let $\overline{\tau} = (\tau + (N_\sigma)_{\mathbb{R}})/(N_\sigma)_{\mathbb{R}}$. The cones $\overline{\tau}$, with $\sigma \prec \tau$, form a fan in $N(\sigma)$, denoted $\mathrm{Star}(\sigma)$ and called the star of σ. We set $V(\sigma) = X(\mathrm{Star}(\sigma))$ the corresponding toric variety. It is a closed subvariety of $X(\Delta)$. For a ray ρ, $V(\rho)$ is a variety of dimension $\dim X(\Delta) - 1$, irreducible and invariant by the action of the torus $T := N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Thus, any ρ in $\Delta(1)$ gives a $T-$Weil divisor, denoted by D_ρ.

For any u of $N^\vee$, we associate a character χ^u given by

$$\begin{cases} T & \to & \mathbb{C}^* \\ v & \mapsto & \langle v, u\rangle \end{cases}$$

If the fan Δ is reduced to a cone, then a $T-$Cartier divisor is the divisor associated to a character χ^u, for some u in $N^\vee$. So, it is of the form $\sum_{\rho\in\Delta(1)} \langle f(\rho), u\rangle D_\rho$, where $f(\rho)$ is the first lattice point along ρ. For a fan Δ not reduced to a cone, to give a $T-$Cartier divisor is to give a vector $u(\sigma)$ on $N^\vee/M^\sigma$, for each cone σ in Δ, such that they agree on overlaps (this is: if $\sigma \prec \tau$, then $u(\tau)$ maps to $u(\sigma)$ via the canonical map from $N^\vee/M^\tau$ to $N^\vee/M^\sigma$). Notice that, if Δ is a simplex with all maximal cones of same dimension, then any Weil divisor is a $\mathbb{Q}-$Cartier divisor. The Euler number of a toric variety $X(\Delta)$ of dimension n is $\#\Delta(n)$, the number of $n-$dimensional cones in Δ.

Notation 1.4. Let G be a finite abelian subgroup of $GL_n(\mathbb{C})$ and let it act on the affine $n-$dimensional space. The Chevalley-Shephard-Todd theorem states that if G is a group generated by pseudo-reflections (i.e. generated by matrices g such that $\mathrm{rank}(g - I_n) = 1$), then the quotient $\mathbb{A}^n/G$ and $\mathbb{A}^n$ are isomorphic. Thus, we can suppose that G is small, i.e., contains no pseudo-reflections. Complex representations of finite groups are semi-simple, so we can choose coordinates on $\mathbb{A}^n$ such that the action becomes diagonal. We fix once for all ε a fixed primitive root of unity of order $r = \#G$. Thus, each element g of G can be identified with a diagonal matrix $\mathrm{diag}(\varepsilon^{a_1}, \dots, \varepsilon^{a_n})$, with $0 \le a_i \le r-1$, for any index i. We associate to such a matrix a vector $\frac{1}{r}(a_1, \dots, a_n) \in \mathbb{Q}^n$, denoted by $v(g)$ or, by abuse, also by g.

Suppose that G is the cyclic group μ_r, acting as above on $\mathbb{A}^n$. This is the same as the action of the cyclic group μ_r on each of the affine lines $A_i = \mathbb{A}^1, i \in \{1, \dots, n\}$, seen as an irreducible one-dimensional eigenspace. Each such action is given by $\mu_r \ni \varepsilon \mapsto (x \mapsto \varepsilon^{a_i} x)$, $0 \le a_i < r$, which is related to the character

$$\begin{cases} \chi_{a_i} : \mu_r & \to & \mathbb{C} \\ \varepsilon & \mapsto & \varepsilon^{a_i} \end{cases}$$

Thus, $\mathrm{diag}(\varepsilon^{a_1}, \dots, \varepsilon^{a_n})$ is a generator of G seen as a subgroup of $GL_n(\mathbb{C})$.

We denote by abuse G by $\frac{1}{\#G}(a_1, \dots, a_n)$ and call the corresponding quotient $\mathbb{A}^n/G$ the abelian quotient singularity $\frac{1}{\#G}(a_1, \dots, a_n)$. A variety having at most quotient singularities is called an orbifold (Fulton) or a $V-$manifold (Satake, Bailey) or a quasi-smooth variety (Steenbrink, Danilov, Kawamata). Following [15], Section 2.2, page 34, a toric variety with simplicial fan is an orbifold. ♣

Example 1.5. We end this section with the example of toric quotient varieties. Recall that $\mathbb{A}^n$ is a toric variety of lattice N_0 and fan reduced to the cone σ_0. Let G be a finite abelian diagonal subgroup of $GL_n(\mathbb{C})$.

With the above notations, let N be the lattice

$$N_0 + \sum_{g \in G} v(g)\mathbb{Z}. \tag{1.1.1}$$

The quotient $\mathbb{A}^n/G$ is a toric variety of lattice N and fan reduced to the cone σ_0.

There are two ways to resolve singularities of a toric quotient variety: either fix the lattice and subdivide the defining fan or refine the lattice and keep unchanged the fan. In the first method, each new ray introduced in the initial fan gives an exceptional divisor on the resolution of the quotient.

For $n = 2$, the method to solve surface cyclic singularities is the **Hirzebruch-Jung algorithm**, recalled here. For more on quotient toric-varieties, see also Section 1.4.2. Let r and t be two non-negative integers with no common divisors and $G = \frac{1}{r}(1, t)$ the cyclic group generated by the matrix $g := \mathrm{diag}(\varepsilon, \varepsilon^t)$, where ε is a fixed primitive root of unity of order $r > 1$. The quotient $\mathbb{A}^2/G$ is a singular toric variety, of lattice $N = \mathbb{Z}^2 + v(g)\mathbb{Z} = \mathbb{Z}^2 + \frac{1}{r}(1, t)\mathbb{Z}$ and fan σ_0.

The algorithm consists in the following steps:

• take the Hirzebruch-Jung continued fraction (with minus) of $r/t = c_1 - \frac{1}{c_2 - \frac{1}{c_3 - \dots}} := [c_1, \dots, c_k]$;

• put $v_0 = (0, 1), v_1 = \frac{1}{r}(1, t)$ and define v_i by $c_i v_i = v_{i-1} + v_{i+1}$ (e.g. $v_{k+1} = (1, 0)$);

• subdivide the cone σ_0 according to the lines passing through the origin and the v_i's; denote by σ_i the cone generated by v_{i-1} and v_i.

Remark that any two consecutive vectors, v_{i-1}, v_i, form a basis of N and the relation between v_{i-1}, v_i and v_{i+1} gives a base change from v_{i-1}, v_i to v_i, v_{i+1} by the $SL_2(\mathbb{Z})$ matrix $\begin{pmatrix} 0 & 1 \\ -1 & c_i \end{pmatrix}$.

The fan of cones $\{\sigma_i\}_{1 \le i \le k+1}$ gives a nonsingular variety, resolution of singularities for $\mathbb{A}^2/G$. Figure 1.1 gives such a fan for the group $1/6(1, 5)$.

The variety thus obtained coincides with the minimal resolution of singularities obtained by blow-ups (see Section 1.1.4). The rays passing by v_i correspond to the exceptional divisors $E_i \simeq \mathbb{P}^1$ with self-intersection numbers $-c_i$.

We call Newton polygon the convex hull (in the positive quadrant) of the division points $(0, 1), 1/r(1, t), \dots, (1, 0)$. In the picture above it is represented by the black thick line connecting v_0 and v_7. ♣

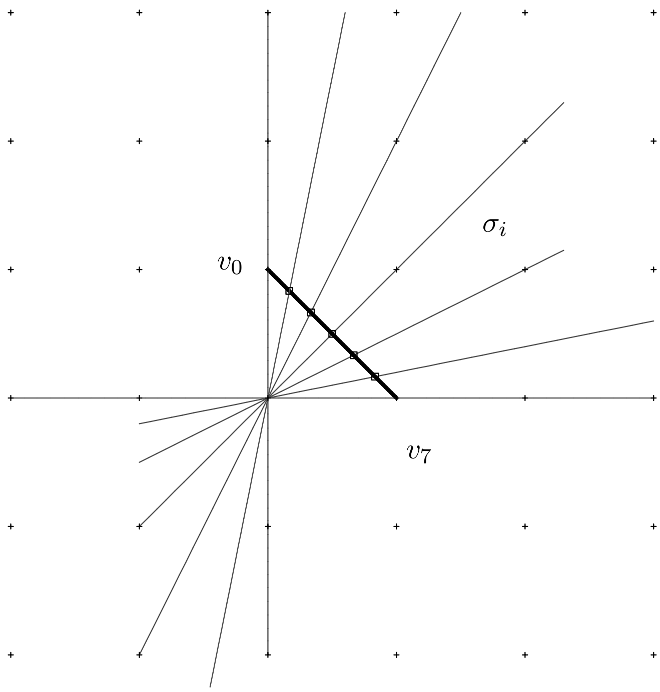

Figure 1.1: Lattice and cones for $1/6(1,5)$.

1.1.2 Crepancy

Remark 1.6. (About the canonical sheaf for singular varieties) For beginning, let us see how to define the canonical sheaf for a singular variety. We follow [37]. We take X a normal variety, with at most canonical singularities and we denote by K_X its canonical divisor. According to [37], Definition 1.1, X has at most canonical singularities if there exist $r \geq 1$ such that rK_X is Cartier and for any resolution $f : Y \to X$ one has:

$$rK_Y = f^*(rK_X) + \sum a_i D_i, \text{with } a_i \text{ some non} - \text{negative integer,} \quad (1.1.2)$$

where $\{D_i\}$ is the set of exceptional prime divisors of f. Let X^{sing} denote the singular locus on X. For a point A in $X \setminus X^{\text{sing}}$, we choose local coordinates $x_1, \ldots, x_n$ at A. A rational canonical differential at A is of the form $f \cdot dx_1 \wedge \cdots \wedge dx_n$, with f in the field of rational functions on $X, k(X)$. We say that this differential is regular at A, if f is regular at A. For a singular point S of X^{sing}, we define a regular differential by asking that there exists an open neighborhood U of S such that, at every point A of $U \cap (X \setminus X^{\text{sing}})$, we have the regularity in the above sense. In other words, we take rational differentials which are regular on the smooth points of a neighborhood of the singular point. This defines the sets of global sections for a sheaf denoted

ω_X and called the canonical sheaf on X. See [37], Section 1.5 for alternative ways of defining ω_X. ♣

Definition 1.7. *([37], Chapter 1, "crepant") Let X be a normal variety with at most canonical singularities and $f : Y \to X$ be a resolution of singularities, with $\{D_i\}$ the set of exceptional prime divisors such that (1.1.2) holds.*

1. *The $\mathbb{Q}-$divisor $\frac{1}{r}\sum a_i D_i, a_i \geq 0$, is called the discrepancy of f.*
2. *We say that the resolution f (respectively Y) is crepant if its discrepancy is zero, i.e. $a_i = 0, \forall i$.*
3. *We say that f (respectively Y) is terminal if $a_i > 0, \forall i$.*

Remark 1.8. Use Remark 1.6 to see that, in terms of sheaves, the crepancy is the same as $f^*(\omega_X) = \omega_Y$. In the particular case of Calabi-Yau varieties (such as quotient toric varieties) crepancy means that the canonical sheaf ω_Y and the structure sheaf $\mathcal{O}_Y$ are isomorphic. ♣

Classical examples of crepant resolutions are the minimal resolution of Du Val surface singularities. We recall such an example in Section 1.1.4.

For a singular variety X of dimension greater than three, the existence of crepant resolutions is rather a difficult problem. A positive answer holds in dimension three: for a finite subgroup G of $SL_3(\mathbb{C})$, the quotient variety $\mathbb{A}^3/G$ admits crepant resolutions. It is interesting to notice (see Section 1.1.3) that the $G-$Hilbert scheme of $\mathbb{A}^3$, $G-$Hilb$\mathbb{A}^3$, is in this case a crepant resolution for $\mathbb{A}^3/G$. For abelian G, the authors of [19] prove by help of Koszul complexes that $G-$Hilb$\mathbb{A}^3$ is a crepant resolution for $\mathbb{A}^3/G$. For G non-abelian, it is known by [7] that the $G-$Hilbert scheme of $\mathbb{A}^3$ is a crepant resolution of singularities for $\mathbb{A}^3/G$.

For a singular X of dimension bigger than four, there are only a few known examples of crepant resolutions. As far as the author knows, there is no general criteria to determine when a singular variety X admits crepant resolutions. An interesting result in this direction can be found in [11]. The authors treats a particular class of toric varieties for which there exist crepant resolutions. The proof is based on reduction to the two-dimensional surface-singularities. More interesting is the case of symplectic groups, solved in [5]. The example we give in this chapter opens new ways of research in this direction. See also the following section for the relation between crepant resolutions and $G-$Hilbert schemes.

1.1.3 About $G-$Hilbert schemes

The notion of $G-$Hilbert scheme was introduced by Y. Ito and I. Nakamura in [20]. It is the $G-$fixed part of a certain $G-$invariant component of the Grothendieck's Hilbert scheme of $\#G$ points. The idea is that the quotient is the space of all the orbits. So, in order to solve the singularity, we need, instead of taking only the set of orbits, to consider a whole collection of smooth $G-$clusters of $\mathbb{A}^n$, of length $\#G$. This is formalized in the notion of $G-$Hilbert scheme.

We recall that (see [16]) the Hilbert scheme associated to a projective scheme X (with ample line bundle $O_X(1)$) is the locally noetherian scheme that represents the functor from the category of schemes to the category of sets:

$$\left\{\begin{array}{lcc} \mathrm{Hilb}_X : (Sch) & \to & (Set) \\ S & \mapsto & \left\{ Z \text{ closed subscheme of} X \times S \;\middle|\; \begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "\pi \text{ flat}"'] & X \times S \arrow[d] \\ & S \end{tikzcd} \right\} \end{array}\right\}.$$

For a point s in S, we denote by $P_s(t) = \chi(O_{\pi^{-1}(s)} \otimes O_X(1)^{\otimes t})$ the corresponding Hilbert polynomial. Because the morphism $\pi : Z \to S$ is flat, if S is connected, the polynomial $P_s(t)$ does not depend on s. This allows to define, for a polynomial P, a sub-functor Hilb_X^P of Hilb_X, as follows. It is the functor sending a scheme S into the family of all closed sub-schemes parameterized by S, having P as Hilbert polynomial. This sub-functor is representable by a noetherian scheme, denoted Hilb_X^P. If the polynomial P is equal to a non-negative integer m, the scheme Hilb_X^m is called the Hilbert scheme of m points on X.

If X is a quasi-projective variety, we view it as an open sub-scheme in a projective variety Y and we consider the corresponding open sub-scheme Hilb_X^P of Hilb_Y^P, this is the scheme parameterizing sub-schemes in X. This allows to define Hilb_X^m for a quasi-projective variety X.

In the sequel, let X be a quasi-projective variety with a faithful action of a finite group G on it. We denote by $\#G$ the order of the group G.

Definition 1.9. *("dynamic" $G-$Hilb, cf. [7], [20] or [36]) Let* $\mathrm{Hilb}_X^{\#G}$ *be the Hilbert scheme of* $\#G$ *points on* X *and let* $(\mathrm{Hilb}_X^{\#G})^G$ *denote its* $G-$*invariant locus by the action of* G*. The "dynamic"* $G-$*Hilbert scheme associated to* X *is the unique irreducible component of* $(\mathrm{Hilb}_X^{\#G})^G$ *corresponding to a general orbit of* G *on* X*.*

Definition 1.10. *("algebraic"* $G-$*Hilb, cf. [9]) The "algebraic"* $G-$*Hilb scheme associated to* X *is the scheme (moduli space) parameterizing all* $G-$*clusters.*

Definition-Notation 1.11. In the the rest of this thesis, for X a quasi-projective variety and G a finite group acting faithfully on it, the $G-$Hilbert scheme of X, denoted $G-\mathrm{Hilb}X$ is the "dynamic" $G-$Hilbert scheme constructed in Definition 1.9. ♣

Remark 1.12. Any free orbit gives a point in the variety $(\mathrm{Hilb}_X^{\#G})^G$. More generally, a point in $(\mathrm{Hilb}_X^{\#G})^G$ is a flat deformation of an orbit. This is why we can consider in Definition 1.9 the irreducible component corresponding to a general orbit. Some authors, consider the scheme $(\mathrm{Hilb}_X^{\#G})^G$ to be the $G-$Hilbert scheme of X. The disadvantage is that it might be reducible. For the "dynamic" $G-$Hilbert scheme, the advantage is that we recover an irreducible scheme, the disadvantage that there is no functor which it represents (as pointed in [10]). ♣

By [7], it is known that for X of dimension three, the dynamic and the algebraic definition of the $G-$Hilbert scheme of X coincide. In the paper [20], the authors prove that, for G a finite subgroup of $SL_2(\mathbb{C})$, the $G-$Hilbert scheme of $\mathbb{A}^2$ coincides with the minimal resolution of du Val singularity $\mathbb{A}^2/G$. It is known, by [37] for example, that that is moreover a crepant resolution. There are natural questions that arise, as follows.

Question 1.13. *Let G be a finite subgroup of $SL_n(\mathbb{C})$ – or $GL_n(\mathbb{C})$, acting faithfully on an affine space $\mathbb{A}^n$.*

Is $G-\mathrm{Hilb}\mathbb{A}^n$ a smooth variety? Is it a resolution of singularities for the quotient $\mathbb{A}^n/G$? A crepant resolution?

A more general form of this question is the following:

Question 1.14. *Let X be a quasi-projective variety and G a finite subgroup of $\mathrm{Aut}(X)$, the group of all automorphisms of X.*

Is $G-\mathrm{Hilb}X$ smooth? When is it a crepant resolution of singularities for X/G?

There are a few positive answers. A main result is related to the McKay correspondence, as in [7]. The idea here is to take the unique irreducible component of $G-\mathrm{Hilb}X$ corresponding to a free orbit; denote it Y. If the condition $\dim Y \times_X Y \leq \dim X$ holds, then one gets a derived equivalence between certain derived categories. This equivalence is called the generalized McKay correspondence. It generalizes the K-theoretical approach of [14] to the language of $D-$theory, by help of Fourier-Mukay transforms. Using this equivalence, it follows that Y is a crepant resolution of singularities for X/G.

In practice, the previous condition does rarely hold. The problem is that the exceptional locus of the Hilbert-Chow morphism $G-\mathrm{Hilb}X \to X/G$ might be big. What is to be wanted is a list of concrete examples where the conjectures above are verified. The most general example is the one of G a finite subgroup of $SL_3(\mathbb{C})$: in this case $G-\mathrm{Hilb}\mathbb{A}^3$ is smooth and it is a crepant resolution of singularities for the quotient $\mathbb{A}^3/G$. For G abelian, there is an algorithm given in [10]. We recall it in Section 1.4.2 and show how to use it for $\mathbb{A}^n$, with $n \geq 4$. There is still an open question how to compute $G-\mathrm{Hilb}\mathbb{A}^3$ for non-abelian G. Some results in this direction are given in [42] – for subgroups of Weyl groups, [29] – for the group H_{168} and in Annexe A – for trihedral groups, based on [28].

Of great interest is the case when the dimension of X is at least four. Nakamura gives in [33] a description of the $G-$Hilbert scheme for the case of G a diagonal subgroup of $SL_n(\mathbb{C})$, for all values of n. This description uses the notion of $G-$graph (see Section 1.2 for a definition) that generalizes the one introduced in [10]. We give in Theorem 1.1 a class of groups for which Nakamura's theorem can be applied and Question 1.13 has a positive answer.

1.1.4 The classical approach on resolutions of singularities

We recall in this section some methods for computing resolutions of singularities. The main method is to blow-up. For the two-dimensional case of Du Val singularities, the resulting variety is also a crepant resolution of the singularities of the initial variety. In general, this is not true. For toric varieties of dimension two, this method can be transposed into the Hirzebruch-Jung algorithm recalled in Section 1.1.1, Example 1.5. A generalization of the above algorithm in dimension three can be found in [10]. We recall in Section 1.4.2 this algorithm and show how to apply it for the groups H_n and why it works in this case.

We give in sequel the classical method of resolving a simple Klein singularity by blow-ups. The disadvantage of this method is that it can not be generalized for the case of a subgroup G of $GL_n(\mathbb{C})$ with $n \geq 3$ acting on $\mathbb{A}^n$ (yet, there is one example by [29]).

Even if the purpose of this chapter is to treat the case of abelian groups, we give here the example of a non-commutative group –the binary dihedral group BD_{4k}. The reason is that similar computations hold for trihedral groups, as shown in Annexe A.

Recall that – for a non-negative integer k – the group BD_{4k} is a finite group generated by two elements σ and τ with relations $\sigma^{2k} = \tau^4 = 1, \sigma^k = \tau^2, \tau\sigma\tau^{-1} = \sigma^{-1}$. This is the semi-direct product of the abelian group generated by σ with the one generated by τ, i.e. $BD_{4k} =< \sigma > \rtimes < \tau >$.

We fix now ϵ a primitive $(2k)^{\mathrm{th}}$ root of unity. Thus, we can identify BD_{4k} with a subgroup of $SL_2(\mathbb{C})$ via the morphism:

$$\sigma \mapsto g = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \tau \mapsto h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

With the above identification, BD_{4k} acts on a natural way on the affine space $\mathbb{A}^2$ by [left] multiplication – matrix $\times$ vector. By geometric invariant theory, the quotient variety, space of all the orbits, $\mathbb{A}^2/BD_{4k}$ is the spectrum of $\mathbb{C}[x,y]^{BD_{4k}}$, the ring of all BD_{4k} invariant polynomials in two variables. It is generated by $u = xy(x^{2k} - y^{2k}), v = x^{2k} + y^{2k}, w = x^2y^2$. This is nothing else but the hypersurface of equation $u^2 - v^2w + 4w^{k+1} = 0$ in $\mathbb{C}^3 = \mathrm{Spec}\mathbb{C}[u,v,w]$ (see Figure 1.2). It has only one isolated singularity – at the origin.

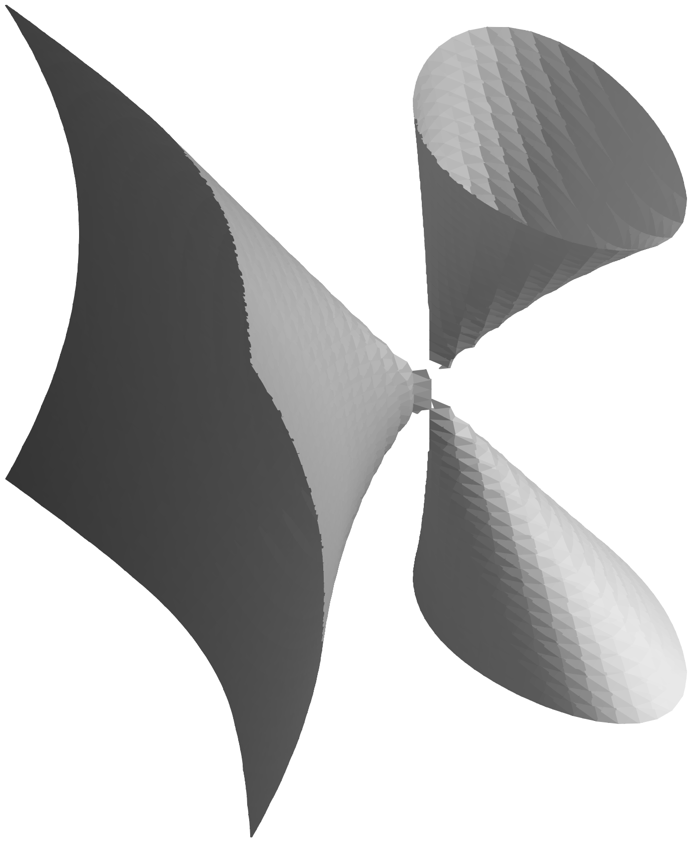

Figure 1.2: Binary dihedral singularity ($k = 2$)

To resolve this singularity, first consider the quotient $\mathbb{A}^2/ < \sigma >$. This is a cyclic singularity of type A_{2k-1}. The corresponding hypersurface is a three dimensional cone of equation $UV = W^{2k}$ in $\mathbb{C}^3$, where one sets this time $U = x^{2k}, V = y^{2k}, W = xy$ (see Figure 1.3). Again, the origin is the only singular point.

The classical method for resolving a singularity of type A_{2k-1} is to blow-up: replace the origin with a projective line $\mathbb{P}^1$ and repeat the procedure until one gets a smooth variety U_k – the minimal resolution of singularities for $\mathbb{A}^2/ < \sigma >$. The exceptional locus on U_k has $2k-1$ curves, $E_1, \ldots, E_{2k-1}$, each isomorphic with a copy of $\mathbb{P}^1$ and with self-intersection -2.

Consider now the action of τ on the quotient $\mathbb{A}^2/<\sigma>$ and also on its minimal resolution of singularities U_k. By this action, the curve E_k is sent to E_k and, for i different from k, the curve E_i is mapped into E_{2k-1-i}. On E_k there are exactly two fixed points – P and Q – that give rise to the singularity on $U_k/<\tau>$. We blow up P and Q. We obtain a smooth variety, denoted $Y_{BD_{4k}}$, that is nothing else but the minimal resolution of $\mathbb{A}^2/BD_{4k}$. The blow-ups of the two points P and Q give two smooth rational curves on $Y_{BD_{4k}}$. We denote by $C_1, \dots, C_k$ the images of $E_1, \dots, E_k$ on $Y_{BD_{4k}}$.

Now, for the case of $X = \mathbb{A}^2/BD_{4k}$, call π the projection map $\mathbb{A}^2 \to \mathbb{A}^2/BD_{4k}$. One wants to see the properties of the canonical sheaf of $\mathbb{A}^2/BD_{4k}$. The affine plane $\mathbb{A}^2$ has $dx \wedge dy$ as a generator for the canonical sheaf. By the action of BD_{4k} on $\mathbb{A}^2$, the matrix g (with the notations above) sends $dx \wedge dy$ to $(\epsilon dx) \wedge (\epsilon^{-1} dy) = \epsilon \cdot \epsilon^{-1} dx \wedge dy = dx \wedge dy$ and for h one gets $-dy \wedge dx = dx \wedge dy$. So $dx \wedge dy$ is invariant under the group action. In order to get a basis for $\omega_{\mathbb{A}^2/BD_{4k}}$, one wants to see this form as a differential form on u, v, w where $X = \mathbb{A}^2/BD_{4k} = \mathrm{Spec}\mathbb{C}[u,v,w]/<u^2 - v^2 w + 4w^{k+1}>$, as above. Put

$$s = \frac{dv \wedge dw}{u}.$$

Differentiating v and w show that $\pi^*(s) = (unit) \cdot dx \wedge dy$, so s is a basis of the canonical sheaf $\omega_{\mathbb{A}^2/BD_{4k}}$ everywhere on $\mathbb{A}^2/BD_{4k} - \{0\}$ in the following sense. At any non-singular point, one can write $\omega_{\mathbb{A}^2/BD_{4k}} = \mathcal{O}_{\mathbb{A}^2/BD_{4k}} \cdot s$, meaning that for $t \in \omega_{\mathbb{A}^2/BD_{4k}}$ one has $t = f \cdot s$, with $f \in \mathbb{C}(\mathbb{A}^2/BD_{4k})$, a regular function. One gets also $\pi^*(\omega_{\mathbb{A}^2/BD_{4k}}) = \omega_{\mathbb{A}^2}$.

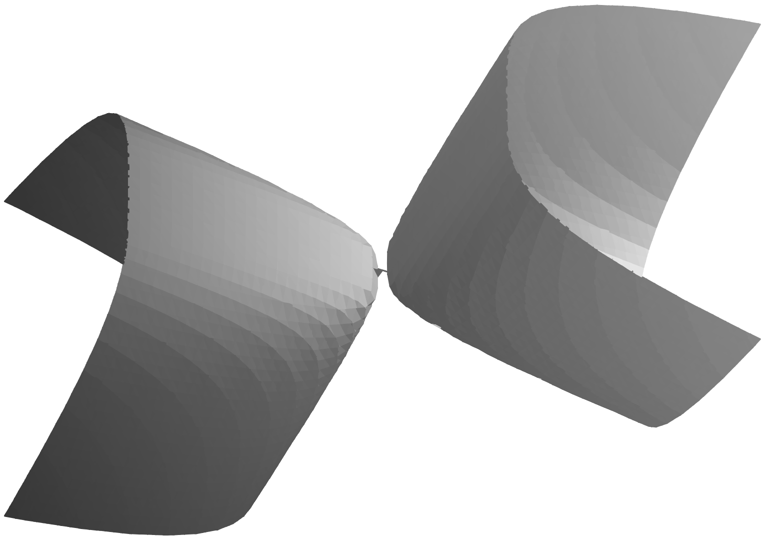

Figure 1.3: Cyclic singularity ($k = 2$)

We remark that $\mathbb{C}(Y_{BD_{4k}}) = \mathbb{C}(\mathbb{A}^2/BD_{4k})$, so s is also a rational differential on $Y_{BD_{4k}}$. A similar computation as the one before proves that s, as a differential on $Y_{BD_{4k}}$, has no poles along the exceptional curves $E_1, \dots, E_{k+2}$, so that it remains regular on the whole minimal resolution and a basis for the canonical sheaf.

These computations show actually that the variety $\mathbb{A}^2/BD_{4k}$ has canonical singularities (in the sense of [37]). Moreover, for the minimal resolution of singularities $F_{BD_{4k}} : Y_{BD_{4k}} \to \mathbb{A}^2/BD_{4k}$, the discrepancy is zero. The result holds also for G a finite subgroup of $SL_2(\mathbb{C})$, acting freely on the affine plane: the quotient $\mathbb{A}^2/G$ has a minimal resolution $F_G : Y_G \to \mathbb{A}^2/G$, obtained by blow-ups. The crepancy property holds: $F_G^*(\omega_{\mathbb{A}^2/G}) = \omega_{Y_G}$.

In higher dimensional cases, it is well-known that minimal resolutions do not make sense and crepant resolutions are looked instead for.

It is then natural to ask: which finite subgroups of $GL_n(\mathbb{C})$ acting on the n dimensional affine space, admit crepant resolutions for the quotient singularity $\mathbb{A}^n/G$? and under which conditions is $G-\mathrm{Hilb}\mathbb{A}^n$ such a crepant resolution?

1.2 $G-$graphs

In the sequel, we fix a non-negative integer n and ε a primitive root of the unity of order $2^n - 1$. We denote by G_n the cyclic group of order $2^n - 1$ generated by ε. We call H_n the cyclic subgroup of $SL_n(\mathbb{C})$ generated by the diagonal matrix $g_n := \mathrm{diag}(\varepsilon, \varepsilon^2, \varepsilon^{2^2}, \dots, \varepsilon^{2^{n-1}})$. We identify this matrix with the vector $h_n := \frac{1}{2^n-1}(1, 2, 4, \dots, 2^{n-1})$. In the literature, the group H_n is also denoted by $\frac{1}{2^n-1}(1, 2, 4, \dots, 2^{n-1})$ (see also Notation 1.4). The aim of this section is to describe all H_n-graphs. The first part contains general definitions and constructions and some examples.

1.2.1 Definitions

In this section, we follow [33] for the definition of a $G-$graph. In the sequel, G is a finite abelian subgroup of $GL_n(\mathbb{C})$. Let $\mathrm{Irr}(G)$ be the set of all irreducible characters of the group G. We use here the notations of Section 1.1.1. We call N_0 the lattice $\mathbb{Z}^n$ with basis $\{e_1, \dots, e_n\}$ and let $\{f_1, \dots, f_n\}$ be its dual basis. We denote by M_0 the additive semigroup generated by 0 and the ${f_i}'$s. We define a semigroup isomorphism from M_0 to the semigroup M of all monomials in n variables (endowed with multiplication), by sending f_i to X_i. Thus, we identify a monomial $X_1^{a_1} \dots X_n^{a_n}$, with a_i non-negative integers, with a vector with n non-negative integer coordinates $(a_1, \dots, a_n)$. More generally, for a "Laurent monomial" $p = X_1^{i_1} \dots X_n^{i_n}$ of $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, we denote by $v(p)$ the vector $(i_1, \dots, i_n)$ of $\mathbb{Z}^n$.

Definition-Notation 1.15. *We denote by Δ_n the set $\{\sum_{i=1}^{n} a_i e_i | \sum_{i=1}^{n} a_i = 1, a_i \geq 0, \forall i\}$. We call it the junior simplex in dimension n. A vector $(v_1, \ldots, v_n)$ such that the sum $\sum_i v_i = 1$ is also called a junior point/vector.*

Example 1.16. For example, in dimension three the junior simplex is nothing else but the triangle of vertices $(1, 0, 0), (0, 1, 0)(0, 0, 1)$. ♣

Definition 1.17. *("age") Let $v := (v_1, \ldots, v_n)$ be a vector of $\mathbb{Q}^n$. We denote by $\operatorname{age}(v)$ the sum of its coordinates and call it the age of v. If $\operatorname{age}(v) = 1$, that is v is in Δ_n, we say that v is junior.*

Definition-Notation 1.18. *Let G be a finite diagonal subgroup of $GL_n(\mathbb{C})$ and identify – as in Example 1.5 – a matrix g of G with a vector $v(g)$ in $\mathbb{Q}^n$. Then, we say that g has age i if the associated vector $v(g)$ has age i. We denote the by $\operatorname{Jun}(G)$ the set $\{v(g) \in \Delta_n | g \in G\}$ and call it the set of junior points of the group G. We denote by $\operatorname{Jun}^+(G)$ the set $\operatorname{Jun}(G) \cup \{e_1, \ldots, e_n\}$ and call it the extended junior set associated to G.*

Notation 1.19. ("weight") We take here H a cyclic group of order r, generated by an element h, and ϵ a fixed root of the unity, of order r. We let H act by algebra automorphism on $\mathbb{C}[X_1, \ldots, X_n]$ by $h \cdot X_k = \epsilon^{a_k} X_k$, with a_k an integer. We then say that H acts by weight a_k on X_k. The weight of a Laurent monomial $p = X_1^{i_1} \ldots X_n^{i_n}$ with respect to the group H is the integer $w(p)$, $0 \leq w(p) \leq r-1$, defined by $\sum_{k=1}^{n} a_k i_k \pmod r$. Here we denote by $\pmod r$ the rest modulo r of an integer.

For example, the group G_n acts by weight 2^{k-1} on X_k. The weight of a Laurent monomial $p = X_1^{i_1} \ldots X_n^{i_n}$ is the unique integer $w(p) \in \{0, \ldots, 2^n - 2\}$, that satisfies $\sum_{k=1}^{n} 2^k i_k \equiv w(p) \pmod{2^n - 1}$. The action of the generating element g_n of the group H_n on $\mathbb{C}[X_1, \ldots, X_n]$ is the same as the weighted action of G_n on $\mathbb{C}[X_1, \ldots, X_n]$, so we also say that the weight of p with respect to the group H_n is $w(p)$. ♣

Definition 1.20. *Given $\chi \in \operatorname{Irr}(G)$ and $p = X_1^{i_1} \ldots X_n^{i_n}$ a [Laurent] monomial, we say that p and χ are associated if we have:*

$$g \cdot p = \chi(g)p, \quad \forall g \in G\,.$$

Definition 1.21. *("$G-$graph", cf. [33], Definition 1.4) A subset Γ of M is called a $G-$graph if the following conditions hold:*

1. *it contains the constant monomial* 1*;*
2. *if p is in Γ and a monomial q divides p, then q is also in Γ;*
3. *the map* $\mathrm{wt} : \Gamma \to \mathrm{Irr}(G)$ *sending a monomial to its associated character (as in Definition 1.20), is a bijection.*

We denote by $\mathrm{Graph}(G)$ the set of all $G-$graphs.

Remark 1.22. Compare the previous definition with Definition 1.4 of [33], given in terms of the semigroup M_0. A similar notion can be defined for some particular classes of non-abelian groups. See Annexe A for the case of trihedral groups. ♣

Example 1.23. By [33], Lemma 1.6, any finite abelian subgroup of $GL_n(\mathbb{C})$ admits a $G-$graph. We consider the particular case when G is the diagonal subgroup of $GL_n(\mathbb{C})$ generated by the matrix $\mathrm{diag}(\epsilon^{a_1}, \ldots, \epsilon^{a_n})$, with ϵ a primitive root of unity, of order r. We suppose moreover that there exists an index k for which $a_k = 1$. Then, the set $\{1, X_k, \ldots, X_k^{r-1}\}$ is a $G-$graph. A $G-$graph containing only t of the given n variables $X_1, \ldots, X_n$, will be called a $G-$graph of order t. For $t = 1$, we say that the $G-$graph is linear and for $t = 2$ we say it is planar. ♣

By condition (3) in Definition 1.21, there is a unique monomial of Γ associated to any character of $\mathrm{Irr}(G)$. Thus, we define a map $\mathrm{wt}_\Gamma : M \to \Gamma$, by sending a monomial to the unique element of Γ with the same associated character.

Definition 1.24. *("ratio") Given a monomial p of M, we call the fraction* $p/\mathrm{wt}_\Gamma(p) \in \mathbb{C}(X_1, \ldots, X_n)$ *the ratio of p with respect to Γ.*

Definition-Notation 1.25. *("rate") Let Γ be a $G-$graph, m a monomial in Γ and p a monomial in $M \setminus \Gamma$. We denote by $r_\Gamma(p, m)$ the number* $\max\{t \in \mathbb{N} | (\mathrm{wt}_\Gamma(p))^t$ *divides* $m\}$ *and call it the rate of p in m [along Γ]. By definition, if* $\mathrm{wt}_\Gamma(p) = 1$*, the rate is zero.*

Construction 1.26. ("deformation"; cf. [33], Definition 2.6, $G-$igsaw transform) Let Γ be a $G-$graph and p a monomial of $M \setminus \Gamma$, or more generally in M. We consider the set $\{m(p/\mathrm{wt}_\Gamma(p))^{r_\Gamma(p,m)} | \, m \in \Gamma\}$. We denote it by $D_p(\Gamma)$ and we call it the deformation of Γ along p. We say that p is the direction of deformation and we call the above operation a deformation process. If the direction of deformation contains only one of the variables, say X_k, we call the deformation a principal deformation in direction k.

In practice, the deformation process consists to replace any occurrence in Γ of the monomial $\mathrm{wt}_\Gamma(p)$ and its powers, by p and its powers. In particular, $\mathrm{wt}_\Gamma(p)$ is replaced by p, so that p belongs to the set $D_p(\Gamma)$. By the deformation process, a monomial m is replaced by a monomial of the same weight. If p is already in Γ the sets $D_p(\Gamma)$ and Γ coincide. ♣

Remark 1.27. We remark that $D_p(\Gamma)$ might not be a $G-$graph. The monomial 1 might not be in $D_p(\Gamma)$. For example, if the direction of deformation is a monomial $p \neq 1$ with $\mathrm{wt}_\Gamma(p) = 1$, then the resulting set $D_p(\Gamma)$ is the set $p\Gamma := \{pm|m \in \Gamma\}$ and it is never a $G-$graph. In terms of weight, this means that if $w(p) = 0$ then $D_p(\Gamma)$ is not a $G-$graph. ♣

Definition 1.28. *("maximal power","power vector") Let S be a non-empty finite set of monomials of $\mathbb{C}[X_1, \dots, X_n]$ and X_k one of the variables. We set $\mathrm{mp}_S(X_k)$ to be $-\infty$ if S doesn't contain any monomial in X_k and $\max\{l \in \mathbb{N}|X_k^l \in S\}$ otherwise. We call it the maximal power associated to the variable X_k in the set S. The vector $\mathrm{pv}(S) := (\mathrm{mp}_S(X_k))_{1\leq k\leq n}$ is called the power vector associated to S.*

Finally, we associate to a $G-$graph, a cone, a semi-group and an ideal as follows. We denote by N the lattice $N_0 + \sum_{g\in G} v(g)\mathbb{Z}$, with $v(g)$ the vector associated to an element g of G (see Section 1.1.1, Example 1.5 and relation (1.1.1)).

Definition-Notation 1.29. *Let Γ be a $G-$graph.*

1. *We define a cone in $\mathbb{R}^n$ by $\sigma(\Gamma) := \{u \in \mathbb{R}^n|\langle u, v(p/\mathrm{wt}_\Gamma(p))\rangle \geq 0, \forall p \in M\}$ and we call it the cone associated to the $G-$graph Γ. We put $\mathrm{Fan}(G)$ to be the set of all the cones $\sigma(\Gamma)$ when Γ runs over $\mathrm{Graph}(G)$.*

2. *We denote by $S(\Gamma)$ the sub-semigroup of $N^\vee$ generated by vectors $v(p/\mathrm{wt}_\Gamma(p))$, where p runs over the set M of all monomials.*

3. *We denote by $V(\Gamma)$ the variety $\mathrm{Spec}(\mathbb{C}[S(\Gamma)])$.*

4. *We call $I(\Gamma)$ the ideal of $\mathbb{C}[X_1, \dots, X_n]$ generated by all the monomials of M that are not in Γ.*

5. *We denote by $T(\Gamma)$ the set $\{v(X_t^{\mathrm{mp}_\Gamma(X_t)+1}/\mathrm{wt}_\Gamma(X_t^{\mathrm{mp}_\Gamma(X_t)+1})) \in \mathbb{Z}^n|t \in \{1, \dots, n\}\}$.*

Remark 1.30. The set $T(\Gamma)$ previously defined can be seen more explicitly as follows. We take each of the variables X_t. We compute $\min\{p \in \mathbb{N}|X_t^p \notin \Gamma\}$, this is actually $\mathrm{mp}_\Gamma(X_t) + 1$. We consider the monomial $X_t^{\mathrm{mp}_\Gamma(X_t)+1}$, search for its associated character and find the unique monomial of Γ with the same associated character, this is $\mathrm{wt}_\Gamma(X_t^{\mathrm{mp}_\Gamma(X_t)+1})$. Then, the set $T(\Gamma)$ is nothing else but the set of vectors in $\mathbb{Z}^n$ associated to Laurent monomials $X_t^{\mathrm{mp}_\Gamma(X_t)+1}/\mathrm{wt}_\Gamma(X_t^{\mathrm{mp}_\Gamma(X_t)+1})$, where t runs over $\{1, \dots, n\}$. ♣

1.2.2 H_n−graphs

This sequel is dedicated to the description of H_n−graphs.

Remark 1.31. We recall that an irreducible character χ_w of H_n, with $w \in \{0, \dots, 2^n - 2\}$, is given by:

$$\begin{cases} \chi_w : H_n & \rightarrow & \mathbb{C}^* \\ g_n & \mapsto & \varepsilon^w. \end{cases}$$

So, a monomial $p = X_1^{i_1} \dots X_n^{i_n}$ is associated to a character χ_w if and only if $w(p)$ equals w. Thus, a H_n−graph is a set of $2^n - 1$ monomials, including the constant monomial 1 and with weights from 0 to $2^n - 2$, satisfying (2) of Definition 1.21. ♣

Lemma 1.32. *Let n be a non-negative integer. Every H_n−graph Γ is of one of the following two types:*

1. *("non-linear H_n−graphs of order $k \in \{1, \dots, n-2\}$") a set of $2^n - 1$ monomials, with power vector*

$$\mathrm{pv}(\Gamma) = (\underbrace{0, \dots, 0}_{b \text{ times}}, 2^{i_1+1} - 1, \underbrace{0, \dots, 0}_{i_1 \text{ times}}, 2^{i_2+1} - 1, \underbrace{0, \dots, 0}_{i_2 \text{ times}}, \dots$$

$$\dots, 2^{i_{n-k}+1} - 1, \underbrace{0, \dots, 0}_{i_{n-k}-b \text{ times}}),$$

where $1 \leq k \leq n-2, 0 \leq i_j \leq k, i_1 + \dots + i_{n-k} = k, 0 \leq b \leq i_{n-k}$. In this case, a monomial m is in Γ if and only if $m \neq \prod_{l=1}^{n} X_l^{\mathrm{mp}_\Gamma(X_l)}$ and $\deg_{X_l}(m) \leq \mathrm{mp}_\Gamma(X_l), \forall l \in \{1, \dots, n\}$.

2. *("linear H_n−graphs, $k = n-1$") a set of $2^n - 1$ monomials, with power vector*

$$\mathrm{pv}(\Gamma) = (\underbrace{0, \dots, 0}_{b \text{ times}}, 2^n - 2, \underbrace{0, \dots, 0}_{n-b-1 \text{ times}}),$$

where $0 \leq b < n$. In this case, a monomial m is in such a H_n−graph if and only if it divides $X_{b+1}^{2^n-2}$.

Proof:

Let Γ be a H_n−graph. If it is a linear one, it is clear that it should be of the form provided by point 2 of the Lemma. We suppose therefore that Γ contains at least two of the variables $X_1, \dots, X_n$. Then, we can suppose that one of those variables is X_t, for some integer $2 \leq t \leq n$. Because X_t occurs effectively in Γ, its maximal power is a positive integer. We argue by contradiction and we suppose that the maximal power of X_t in Γ is different from any integer $2^{i+1}-1$, for $i \in \{0, \ldots, n-1\}$.

First, we remark that if $2^{n-1}-1 < \mathrm{mp}_\Gamma(X_t) < 2^n - 2$, then none of the variables X_i, for $1 \leq i \leq n, i \neq t$, can occur in Γ. This is because the monomials $X_t^{2^j}$ have to be in Γ for any $j, 0 \leq j \leq n-1$, by condition 2 of Definition 1.21. So no monomial of weight $2^j, 1 \leq j \leq n$, can occur in Γ, following Remark 1.31. But in this case the set Γ would have less then $2^n - 1$ monomials which is a contradiction.

We suppose now that $2^l - 1 < \mathrm{mp}_\Gamma(X_t) < 2^{l+1} - 1$, for some integer $l \in \{1, \ldots, n-2\}$. Without loss of generality, we can assume that $l+t+1 \leq n$, the other possibility can be treated similarly. As above, by an argument on the weights, we see that only monomials in $\{X_1, \ldots, X_{t-1}, X_{l+t+1}, \ldots, X_n\}$ can occur in Γ.

We need in Γ a monomial m of weight $[(\mathrm{mp}_\Gamma(X_t)+1)2^{t-1}] \pmod{(2^n-1)}$. We consider the case $[(\mathrm{mp}_\Gamma(X_t)+1)2^{t-1}] < (2^n-1)$, that is $l \leq n-t$, the other case can be treated similarly.

If the monomial X_t^p for some positive p occurs in m, then the two monomials $X_t^{\mathrm{mp}_\Gamma(X_t)+1-p}$ and m/X_t^p are in Γ and have the same weight, which is a contradiction.

Thus, we can take the monomial m of the form

$$X_1^{a_1} \ldots X_{t-1}^{a_{t-1}} X_{l+t+1}^{a_{l+t+1}} \ldots X_n^{a_n},$$

for some non-negative integers a_i. Then, there exists a non-negative integer v such that

$$a_1+2a_2+\cdots+2^{t-2}a_{t-1}+\cdots+2^{l+t}a_{l+t+1}+\cdots+2^{n-1}a_n = (\mathrm{mp}_\Gamma(X_1)+1)2^{t-1}+v(2^n-1). \tag{1.2.1}$$

In the sequel, we treat the case $v = 0$. We consider then that we have the equality

$$a_1+2a_2+\cdots+2^{t-2}a_{t-1}+\cdots+2^{l+t}a_{l+t+1}+\cdots+2^{n-1}a_n = (\mathrm{mp}_\Gamma(X_1)+1)2^{t-1}.$$

We write this as

$$a_1 = -(2a_2+\cdots+2^{t-2}a_{t-1}+\cdots+2^{l+t}a_{l+t+1}+\cdots+2^{n-1}a_n)+(\mathrm{mp}_\Gamma(X_1)+1)2^{t-1}.$$

We deduce that a_1 has to be divisible by two. If we suppose that a_1 is not zero, then it should be at least two. But this means that no monomial of weight two other then X_1^2 can occur in Γ. Thus, a_2 is zero. Then, we deduce that a_1 is divisible by four and we conclude that monomial X_3 doesn't occur in Γ. So the integer a_3 is also zero. Thus, recursively, we obtain that $a_2 = \cdots = a_{t-1}$. Then, a_1 should be divisible by 2^{t-1}. Which means in particular that $X_1^{2^{t-1}}$ and X_t both occur in Γ and have the same weight. This contradicts the definition of a H_n-graph. So we conclude that a_1 is zero.

Thus, the equality reduces to:

$$a_2 + \cdots + 2^{t-3}a_{t-1} + \cdots + 2^{l+t-1}a_{l+t+1} + \cdots + 2^{n-2}a_n = (\mathrm{mp}_\Gamma(X_1) + 1)2^{t-2}.$$

A similar argument as above shows that a_2 has to be zero. Recursively, we find that all integers $a_i, 1 \leq i \leq t-1$, are zero. Bur then, we have

$$2^{l+t}(a_{l+t+1} + \cdots + 2^{n-l-t-1}a_n) = (\mathrm{mp}_\Gamma(X_1) + 1)2^{t-1},$$

which leads to

$$2^{l+1}(a_{l+t+1} + \cdots + 2^{n-l-t-1}a_n) = (\mathrm{mp}_\Gamma(X_1) + 1). \tag{1.2.2}$$

In particular, because $\mathrm{mp}_\Gamma(X_1) + 1$ is positive, we deduce that also the left hand side is positive, more precisely it is at least 2^{l+1}. By hypothesis, the right hand side is less then $2^{l+1} - 1$. This is a contradiction.

We give an idea of the proof for the general case, that is $v > 0$. We consider an equality of the form (1.2.1), for some non-negative coefficients a_i.

As before, we deduce that $(a_1 + v)$ is divisible by two. This implies that a_1 and v should be both either even, or odd. We consider the case when both are even, the other one being similar. The case $a_1 = v = 0$ is the one we proved before, but with a_2 as a first term instead of a_1. If a_1 is zero, then v is even. We subdivide (1.2.1) by two and we obtain an equality where this time the first term is a_2, instead of a_1. Similar considerations as the one bellow can apply, so we don't treat separately this case.

We are now in the situation when a_1 is an integer greater then two. If $a_1 = 2$, then an analysis on the weights shows that X_2 can not occur in Γ, which implies in particular that a_2 is zero. This means that $(a_1 + v)$ is divisible by four. This implies that v is of the form $4v' - 2$ for some positive v'. Then, subdivide (1.2.1) by two and obtain

$$-1 + 2v' + 2a_3 + \cdots + 2^{t-3}a_{t-1} + 2^{l+t-1}a_{l+t+1} + \cdots + 2^{n-2}a_n = (\mathrm{mp}_\Gamma(X_1) + 1)2^{t-2} + v2^{n-1}.$$

This is the equality between an even and an odd number, which is impossible. Thus, we conclude that a_1 is at least four, which in particular gives $a_3 = 0$, following that X_3 can not occur in Γ. A recurrent argument shows that all integers $a_i, 1 < i < t$, are zero.

This leads to an equality similar to (1.2.2) and thus to a contradiction, by help of the hypothesis $2^l < 1 + \mathrm{mp}_\Gamma(X_t) < 2^{l+1}$.

Up to now, we proved that for any $t \geq 2$, the maximal power of the variable X_t in a H_n−graph is of the form $2^i - 1$, for some non-negative i.

A similar proof holds also for the case when X_1 occurs effectively in Γ. As above, we argue by contradiction. If $2^{n-1} - 1 < \mathrm{mp}_\Gamma(X_1) < 2^n - 2$, then, arguing on the weights, we conclude that none of the variables X_i, for

$2 \le i \le n$, can occur in Γ, which is impossible. So, we can suppose that $2^l - 1 < \mathrm{mp}_\Gamma(X_1) < 2^{l+1} - 1$, for some integer $l \in \{1, \ldots, n-2\}$. Then, the only monomials that can occur in Γ are $\{X_1, X_{l+1}, \ldots, X_n\}$. Arguing as before, there is no monomial of weight $\mathrm{mp}_\Gamma(X_1) + 1$ in Γ. We conclude that the maximal power of X_1 is also of the form $2^i - 1$, for some non-negative i.

Let X_t be one of the variables that effectively occur in Γ and let $2^i - 1$ be its maximal power, for some positive integer i. Without loss of generality, we can suppose that $i + t < n$. The weight 2^{t+i} can be given but by a monomial of the form $X_s^{2^q}$, for some positive s and q. It is clear that s is different from t. If $s < t$, then $q = t + i - s + 1$. This is a contradiction because $X_s^{2^{t-s+1}}$ and X_t have the same weight and are both in Γ. Let now s be greater than t. Arguing on the weights, we see that none of the variables $X_{t+l}, 1 \le l \le i-1$, can occur in Γ. We conclude that $s > t + i$. If $s \ne t + i + 1$, as above $X_s^{2^{q-1}}$ and X_t are both in Γ and have the same weight, which is a contradiction. We conclude that X_{t+i+1} has to be in Γ.

Thus, the power vector of an H_n−graph Γ is of the form

$$\mathrm{pv}(\Gamma) = (\underbrace{0, \ldots, 0}_{b \text{ times}}, 2^{i_1+1} - 1, \underbrace{0, \ldots, 0}_{i_1 \text{ times}}, 2^{i_2+1} - 1, \underbrace{0, \ldots, 0}_{i_2 \text{ times}}, \ldots$$

$$\ldots, 2^{i_{n-k}+1} - 1, \underbrace{0, \ldots, 0}_{i_{n-k} - b \text{ times}}),$$

for some k, $1 \le k \le n-2$ and non-negative integers b and $i_j, 1 \le j \le n-k$. We notice that $b > 0$ means that X_1 doesn't occur in Γ. The maximal power $2^{i_j+1} - 1$, $1 \le j \le n-k$, corresponds to the variable $X_{b+j+i_1+\cdots+i_{j-1}}$. In particular, the last non-zero position of the power vector of Γ is $b + n - k + i_1 + \cdots + i_{n-k-1}$. We deduce that $n = (b + n - k + i_1 + \cdots + i_{n-k-1}) + (i_{n-k} - b)$, which implies that $i_1 + \cdots + i_{n-k} = k$.

The assertion on the form of the monomials of an H_n−graph Γ follows easily, noticing that the monomial $\prod_{l=1}^{n} X_l^{\mathrm{mp}_\Gamma(X_l)}$ has weight one, therefore is not in Γ. This ends the proof. ■

Notation 1.33. 1. We call the integer b of the Lemma 1.32, (1), the skip of the corresponding H_n−graph.

2. Let n be a non-negative integer and $v = (v_1, \ldots, v_n)$ be a vector of length n. Let $1 \le i \le n$ and $0 \le t \le n - i$ be two integers. We call the vector $(v_i, \ldots, v_{i+t})$ a range of v of length $t + 1$ and origin i. In particular, for Γ an H_n−graph, the power vector $\mathrm{pv}(\Gamma)$ is formed with ranges of the form $(2^i - 1, 0, \ldots, 0)$, for some positive i.

♣

Remark 1.34. In the above Lemma, the passage from an H_n−graph Γ of order k and power vector $\mathrm{pv}(\Gamma) = (\underbrace{0,\ldots,0}_{b \text{ times}}, 2^{i_1+1}-1, \underbrace{0,\ldots,0}_{i_1 \text{ times}}, 2^{i_2+1}-1, \underbrace{0,\ldots,0}_{i_2 \text{ times}}, \ldots, 2^{i_{n-k}+1}-1, \underbrace{0,\ldots,0}_{i_{n-k}-b \text{ times}})$ to some $(k+1)^{\mathrm{th}}$ order H_n−graph consists of gluing two neighboring ranges, as follows. We take some range $(2^{i+1}-1, \underbrace{0,\ldots,0}_{i \text{ times}})$ of $\mathrm{pv}(\Gamma)$, with origin p, where $i \in \{i_1, \ldots, i_{n-k}\}$. This means that the variable X_p occurs in Γ with maximal power $2^{i+1}-1$, that the variable X_{p+i+1} is also in Γ and we let $(2^{j+1}-1, \underbrace{0,\ldots,0}_{j \text{ times}})$ be its corresponding range in $\mathrm{pv}(\Gamma)$. The monomial $X_p^{2^{i+1}}$ is not in Γ and we have $\mathrm{wt}_\Gamma(X_p^{2^{i+1}}) = X_{p+i+1}$. Deforming along $X_p^{2^{i+1}}$ gives a new H_n−graph where the range $(2^{i+1}-1, \underbrace{0,\ldots,0}_{i \text{ times}})$ concatenates with the range $(2^{j+1}-1, \underbrace{0,\ldots,0}_{j \text{ times}})$ to give $(2^{i+j+2}-1, \underbrace{0,\ldots,0}_{i+j+1 \text{ times}})$. For example, for the range $(2^{i_1+1}-1, \underbrace{0,\ldots,0}_{i_1 \text{ times}})$ of origin $b+1$, deforming along the principal direction $X_b^{2^{i_1+1}}$ gives the H_n−graph of power vector $(\underbrace{0,\ldots,0}_{b \text{ times}}, 2^{i_1+i_2+1}-1, \underbrace{0,\ldots,0}_{i_1+i_2+1 \text{ times}}, \ldots, 2^{i_{n-k}+1}-1, \underbrace{0,\ldots,0}_{i_{n-k}-b \text{ times}})$ and order $k+1$. We remark that Γ has $n-k$ ranges and that the newly obtained H_n−graph has $n-(k+1)$ ranges, that is we pass from an H_n−graph of order k to an H_n−graph of order $k+1$.

We also have the reverse operation of concatenating ranges, this is breaking a range to obtain a H_n−graph. We use the same notations as above: we take an H_n−graph Γ and a range $(2^{i+1}-1, \underbrace{0,\ldots,0}_{i \text{ times}})$. If t is an integer between 1 and i, the monomial X_{p+t} is not in Γ. It has the same weight as $X_p^{2^t}$ which is in Γ. So, we can deform Γ by replacing $X_p^{2^t}$ and its powers with X_{p+t}. We obtain an H_n−graph of order $k-1$ where the range $(2^{i+1}-1, \underbrace{0,\ldots,0}_{i \text{ times}})$ is broken into two ranges $(2^t-1, \underbrace{0,\ldots,0}_{t-1 \text{ times}})$ and $(2^{i+1-t}-1, \underbrace{0,\ldots,0}_{i-t \text{ times}})$. This gives a new H_n−graph of order $k-1$ because the sum of all i_j decreases by one, following $(t-1)+(i-t) = i-1$.

To conclude, there are two ways to compute $\mathrm{Graph}(H_n)$. We can start with the H_n−graph of power vector $(\underbrace{1,\ldots,1}_{n \text{ times}})$ and use the method of concatenating ranges. We end by finding the linear H_n−graphs. Symmetrically, we can start with a linear H_n−graph and apply the reverse process of breaking ranges. This leads from an H_n−graph of order k to an H_n−graph of order $k-1$. The process stops while obtaining the H_n-graph of power vector $(1,\dots,1)$, n occurences of 1. ♣

Example 1.35. Let us see how to apply Lemma 1.32 and Remark 1.34 to compute all the H_n-graphs for $n=3$. In this case, the group H_3 has order $2^3-1=7$. We start with the H_3-graph of power vector $(1,1,1)$, that is $\Gamma_1 := \{1, X_1, X_2, X_3, X_1X_2, X_1X_3, X_2X_3\}$. By concatenating two neighboring ranges, we obtain the H_3-graphs $\Gamma_2, \Gamma_3, \Gamma_4$ of power vectors respectively $(1,3,0)$, $(3,0,1)$ and $(0,1,3)$.

For example, the first H_3-graph Γ_2 corresponds to a deformation of the initial H_3-graph in the direction X_2^2 and consists in replacing X_3 by X_2^2 in Γ. Thus, the set Γ_2 is nothing else but $\{1, X_1, X_2, X_2^2, X_1X_2, X_1X_2^2, X_2^3\}$. We deform once more $\Gamma_2, \Gamma_3, \Gamma_4$ to obtain the linear H_3-graphs $\Gamma_5, \Gamma_6, \Gamma_7$ of power vectors $(0,6,0)$, $(6,0,0)$ and $(0,0,6)$. For example, Γ_5 with power vector $(0,6,0)$ is obtained from Γ_2 by deforming along X_2^4. The whole deformation process and the corresponding ratios are shown in Figure 1.4 bellow. The two-headed arrows show that actually the process of concatenating ranges has a reverse: the breaking-ranges process. We remark also that a ratio $m:n$, for some monomials m and n, can be considered in two ways.

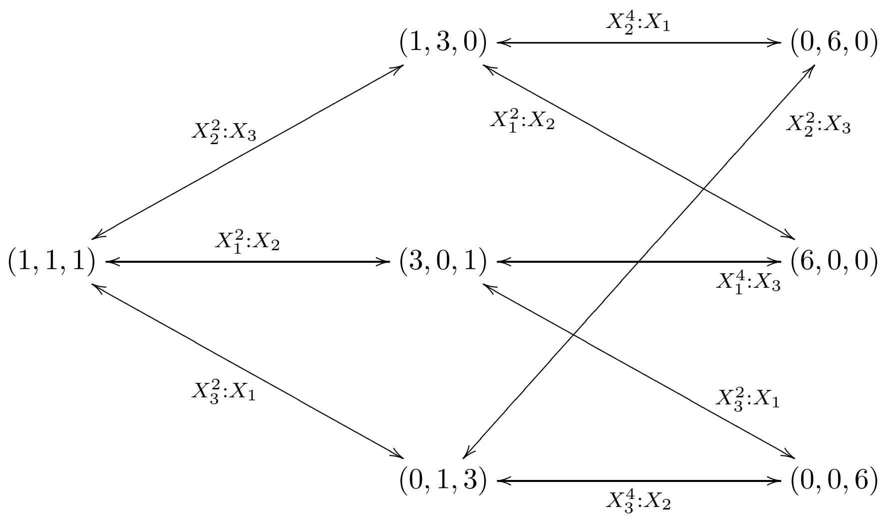

Figure 1.4: Deforming process for GraphH_3.

From left to right it corresponds to the concatenating process and deformation along the denominator n; from right to left it corresponds to the breaking process and deformation along the numerator m. ♣

Corollary 1.36. *The number of H_n-graphs is the cardinal of H_n, this is:* $\#\mathrm{Graph}(H_n) = 2^n - 1$.

Proof:

The number of linear H_n-graphs is n. We compute the number of H_n-graphs at order k, $0 \leq k \leq n-2$. There are two types of such graphs, those with zero-skip and those with non-zero skip. Let us denote by $N(n,k)$ the number of non-linear H_n-graphs of order k with skip $b = 0$. We want to express the number of H_n-graphs of non-zero skip as a function of $N(\cdot,\cdot)$.

We make the following remark. Let Γ be a non-linear H_n-graph of order k, zero-skip and power vector v of the form:

$$(2^{i_1+1} - 1, 0, \ldots 0, 2^{i_{n-k}+1} - 1, 0, \ldots, 0),$$

for some non-negative indices i_j. We denote the integer i_{n-k} by l, such that the last range of the above power vector is of the form $(2^{l+1} - 1, \underbrace{0, \ldots, 0}_{l \text{ times}})$.

We fix an integer t, $1 \leq t \leq l$, and proceed to a cyclic permutations of v to the right with t places. Denote by v_t^l the new vector thus obtained. This vector is nothing else but the power vector of a non-linear H_n-graphs of order k and non-negative skip $b = t$:

$$(\underbrace{0, \ldots, 0}_{b \text{ times}}, 2^{i_1+1} - 1, 0, \ldots 0, 2^{i_{n-k}+1} - 1, 0, \ldots, 0).$$

Now, there are l possibilities of cyclic permutations to get a power vector of the form v_t^l. So, from a given H_n-graph with zero-skip and last range of length $l+1$, we obtain l non-linear H_n-graphs of order k and positive skip. Furthermore, all non-linear H_n-graphs of order k and positive skip are obtained this way exactly once.

Let us see now how many such H_n-graphs with zero-skip and last range of length $l+1$ are. Such an H_n-graph has a power vector with a total of $n-k$ ranges from which we already know one. Because we also know the last $l+1$ positions – that is $(2^{l+1} - 1, \underbrace{0, \ldots, 0}_{l \text{ times}})$ – there are only $n-(l+1)$ positions to be filled in.

After removing from v the range $(2^{l+1} - 1, \underbrace{0, \ldots, 0}_{l \text{ times}})$, by Lemma 1.32, the resulting vector corresponds to a power vector of length $n-(l+1)$ and a total of $n-k-l$ ranges. We write

$$n - k - 1 = [n - (l+1)] - [(k+1) - (l+1)] = [n - (l+1)] - (k - l).$$

Thus, we have a total of $N(n-(l+1), k-l)$ power vectors with zero-skip and last range of length $l+1$.

So, by cyclic permutation we get a total of $lN(n-(l+1),k-l)$ corresponding power vectors of non-zero-skip. Now, l varies from 1 to k, so we get that the total number of possible power vectors of non-zero-skip at order k is $\sum_{l=1}^{k} lN(n-(l+1),k-l)$. So, the total number of H_n-graphs at order k is $N(n,k)+\sum_{l=1}^{k} lN(n-(l+1),k-l)$.

We compute $N(n,k)$, for some k less than $n-2$. In other words, we want to find how many possibilities are to fill-in n positions with $n-k$ ranges of the form $(2^{i+1}-1,\underbrace{0,\dots,0}_{i \text{ times}})$, for some appropriate i. But, the first position is never zero, so this is the same as to fill in $n-1$ positions. Also, we have the choice for only $n-k-1$ ranges, because the first range is always filled in by some vector of the form $(2^{i_1+1}-1,\underbrace{0,\dots,0}_{i_1 \text{ times}})$, for $i_1\geq 0$. We obtain $N(n,k)=C_{n-1}^{n-k-1}=C_{n-1}^{k}$.

So, in general, the number of H_n-graphs of order k is given by $C_{n-1}^{k}+\sum_{l=1}^{k} lN(n-(l+1),k-l)=C_{n-1}^{k}+\sum_{l=1}^{k} lC_{n-(l+1)-1}^{k-l}=C_{n-1}^{k}+\sum_{l=1}^{k} lC_{n-l-2}^{k-l}=C_{n-1}^{k}+C_{n-1}^{k-1}=C_{n}^{k}$. Using this result, we find that $\#\mathrm{Graph}(H_n)$ the number of H_n-graphs is 2^n-1. ■

1.2.3 $G-$graphs and $G-$Hilbert schemes

We notice that Definition 1.21 of Section 1.2.1 is a transcription in terms of monomials of Definition 1.4 of [33]. Thus, we can reformulate some of the results of [33] (especially Theorem 2.11) as follows.

Lemma 1.37. *The varieties $V(\Gamma),\Gamma\in\mathrm{Graph}(H_n)$, of Definition-Notation 1.29,(3) can be glued together into a variety denoted $V(\mathrm{Graph}(H_n))$.*

Proof:

Following Remark 1.34, we can compute $\mathrm{Graph}(H_n)$ from a given H_n-graph by deformations along principal directions. Each deformation from a H_n-graph Γ to an H_n-graph Γ' is given by a ratio R (see also Example 1.35). In particular, we get $\mathbb{C}[S(\Gamma)][R]\simeq\mathbb{C}[S(\Gamma')][1/R]$. ■

Theorem 1.38. *Let H_n be the finite abelian subgroup of $SL_n(\mathbb{C})$ generated by the diagonal matrix:*

$$g_n:=\begin{pmatrix} \varepsilon & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon^2 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 0 & \varepsilon^{2^{n-1}} \end{pmatrix},$$

where ε is a primitive root of unity of order $2^n - 1$.

Let $\mathrm{Graph}(H_n)$ *be the set of all H_n–graphs,* $V(\mathrm{Graph}(H_n))$ *the corresponding variety. Let* $\mathrm{Fan}(H_n)$ *be the set of all cones $\sigma(\Gamma)$, when Γ runs over* $\mathrm{Graph}(H_n)$. *Then:*

1. *The set* $\mathrm{Graph}(H_n)$ *can be obtained from a given H_n–graph by deformations,*

2. *The variety* $V(\mathrm{Graph}(H_n))$*, is isomorphic to the H_n–Hilbert scheme of $\mathbb{A}^n$ and it is projective over A^n/H_n.*

3. *The toric variety* $X(\mathrm{Fan}(H_n))$ *is the normalization of H_n–Hilb$\mathbb{A}^n$.*

Proof:

Point (1) is nothing else but Remark 1.34. Parts (2) and (3), are consequences of [33], Theorem 2.11, (iii) and (iv). ■

1.3 The proof

In the sequel, we denote by h_n the vector $\frac{1}{2^n-1}(1, 2, 2^2, \dots, 2^{n-1})$ associated to the diagonal generator of H_n, $g_n := \mathrm{diag}(\varepsilon, \varepsilon^2, \dots, \varepsilon^{2^{n-1}})$, with ε fixed primitive root of unity of order $2^n - 1$.

Notation 1.39. For t an integer in $\{0, 1, 2, \dots, 2^n - 2\}$ we denote by $t \star h_n$ the vector with the l^{th} coordinate equal to $\frac{1}{2^n-1} \times [(2^{l-1}t) \pmod{(2^n-1)}]$, the product between the rest modulo $2^n - 1$ of $2^{l-1}t$ and the fraction $\frac{1}{2^n-1}$. Remark that any value $(2^{l-1}t) \pmod{(2^n-1)}$ is an integer between 1 and $2^n - 2$. We also remark that the vector $2^i \star h_n$, for i in $\{0, 1, 2, \dots, n-1\}$, is nothing else but the vector associated to the matrix $g_n^{2^i}$. We have $2^i \star h_n = \frac{1}{2^n-1}(2^i, 2^{i+1}, \dots, 2^{n-1}, \underbrace{1}_{(n-i+1)^{\text{th}} \text{ position}}, 2, 2^2, \dots, 2^{i-1})$, that is a cyclic permutation of the coordinates of the vector h_n by i positions to the left. The l^{th} position of such a vector is given by $\frac{1}{2^n-1} 2^{(l-1+i) \pmod{n}}$. We remark also that if $i \neq j$ are two integers in $\{0, 1, \dots, n-1\}$, then $2^i \star h_n \neq 2^j \star h_n$. ♣

Remark 1.40. Any integer t between 0 and $2^n - 2$ can be written in the form $2^{m_1(t)} + 2^{m_2(t)} + \dots + 2^{m_{l(t)}(t)}$, with $1 \le l(t) \le n-1$ and $0 \le m_1(t) < m_2(t) < \dots < m_{l(t)}(t) \le n-1$. ♣

We prove here Theorem 1.1. We subdivide the proof into several lemmas.

Lemma 1.41. *For t between 0 and 2^n-2, we have $\mathrm{age}(t\star h_n)=l(t)$, where $l(t)$ is defined in Remark 1.40.*

Proof:

We have the equality $t\star h_n = 2^{m_1(t)}\star h_n + 2^{m_2(t)}\star h_n+\cdots+2^{m_{l(t)}(t)}\star h_n$. We justify this as follows. Let s be any coordinate, $1\le s\le n$. It is enough to show that the sum

$$2^{(s-1+m_1(t)) \pmod n} + 2^{(s-1+m_2(t)) \pmod n} + \cdots + 2^{(s-1+m_{l(t)}(t)) \pmod n}$$

is an integer between 0 and 2^n-1. Following the definition of the integers $m_i(t)$ and the fact that $l(t)\le n-1$, we deduce that this sum is strictly inferior to:

$$\begin{aligned}2^{(s-1) \pmod n} + 2^{(s-1+1) \pmod n} + \cdots + 2^{(s-1+n-1) \pmod n} &= \\ = 2^{s-1}+\cdots+2^{n-1}+1+\cdots+2^{s-2} &= 2^n-1\end{aligned}$$

Each of the vectors $2^{m_i(t)}\star h_n$ is a cyclic permutation of h_n, so it has the same age as h_n. We conclude that $\mathrm{age}(t\star h_n)=\mathrm{age}(h_n)\times l(t)=1\times l(t)=l(t)$.
■

Corollary 1.42. *We have:*

$$\mathrm{age}(t\star h_n)=1 \iff t \textit{ is of the form } 2^i, \textit{ for some } i\in\{0,1,\dots,n-1\}.$$

In particular, $\mathrm{Jun}(H_n)=\{2^i\star h_n | i\in\{0,1,\dots,n-1\}\}$. ■

Lemma 1.43. *Let Γ be a H_n-graph and X_t one of the variables. Then, $\mathrm{wt}_\Gamma(X_t^{\mathrm{mp}_\Gamma(X_t)+1})=X_{q(t)}^{r(t)}$, where:*

1. *$q(t)=(t+\log_2(\mathrm{mp}_\Gamma(X_t)+1)) \pmod n$ and $r(t)=1$, if X_t occurs in Γ — this is if $\mathrm{mp}_\Gamma(X_t)\neq 0$;*
2. *$q(t)=\max\{i \mid 0<i<t, \mathrm{mp}_\Gamma(X_i)\neq 0\}$ and $r(t)=2^{t-q(t)}$ if X_t doesn't occur in Γ and $\{i\mid 0<i<t,\mathrm{mp}_\Gamma(X_i)\neq 0\}\neq\varnothing$;*
3. *$q(t)=\max\{i \mid i\le n, \mathrm{mp}_\Gamma(X_i)\neq 0\}$ and $r(t)=2^{t+n-q(t)}$ if $\mathrm{mp}_\Gamma(X_t)=0$ and $\{i\mid 0<i<t,\mathrm{mp}_\Gamma(X_i)\neq 0\}=\varnothing$.*

In particular, for Γ a linear H_n-graph only on X_i, we have $\mathrm{wt}_\Gamma(X_i^{2^n-1})=1$ and for any $t\neq i$ we have $\mathrm{wt}_\Gamma(X_t)=X_i^{2^{(n+t-i) \pmod n}}$, that is $q(t)=i$ and $r(t)=2^{(n+t-i) \pmod n}$.

Proof:

Let t be an integer between 1 and n. If $\mathrm{mp}_\Gamma(X_t) \neq 0$, the variable X_t effectively occurs in Γ. Then, by Lemma 1.32, we have $\mathrm{mp}_\Gamma(X_t) + 1 = 2^{i+1}$, for some integer i among $i_1, \ldots, i_{n-k}$. The weight of monomial $X_t^{\mathrm{mp}_\Gamma(X_t)+1}$ is $[(\mathrm{mp}_\Gamma(X_t)+1)\times 2^t] \pmod{2^n-1} = 2^{t+i+1} \pmod{2^n-1}$. This means that the corresponding monomial of the same weight in Γ is X_{t+i+1} if $t+i+1 \leq n$, respectively $X_{(t+i+1) \pmod n}$ for $t+i+1 > n$. We conclude that in this case $\mathrm{wt}_\Gamma(X_t^{\mathrm{mp}_\Gamma(X_t)+1}) = X_{q(t)}$, with $q(t) = (t+i+1) \pmod n$, where $i = \log_2(\mathrm{mp}_\Gamma(X_t)+1) - 1$.

Now, if $\mathrm{mp}_\Gamma(X_t) = 0$, X_t doesn't occur in Γ. We consider the set $\{i \mid 0 < i < t, \mathrm{mp}_\Gamma(X_i) \neq 0\}$. If this set is not empty, we denote its maximum by $q(t)$. Then, in Γ there is a range of the form $(2^{i+1}-1, 0, \ldots, 0)$, with origin $q(t)$ and length $i+1$, such that on position $t - q(t)$ of the range we have the zero corresponding to the variable X_t. We notice that $2^{t-q(t)}$ is less then $\mathrm{mp}_\Gamma(X_{q(t)}), i+1$ because $t - q(t) \leq i$. The weight of $X_t^{\mathrm{mp}_\Gamma(X_t)+1} = X_t$ is 2^t and the corresponding monomial of Γ is $X_{q(t)}^{2^{t-q(t)}}$. Thus, we have $\mathrm{wt}_\Gamma(X_t^{\mathrm{mp}_\Gamma(X_t)+1}) = X_{q(t)}^{t-q(t)}$.

If the set $\{i \mid 0 < i < t, \mathrm{mp}_\Gamma(X_i) \neq 0\}$ is empty, this means that the H_n−graph Γ has non-negative skip $b > 0$. Let $q(t)$ be the largest integer less then n such that $\mathrm{mp}_\Gamma(X_{q(t)}) \neq 0$. Then, the power vector of Γ is of the form:

$$(\underbrace{0, \ldots, 0}_{b \text{ times}}, 2^{i_1+1} - 1, \underbrace{0, \ldots, 0}_{i_1 \text{ times}}, \ldots, 2^{i_{n-k}+1} - 1, \underbrace{0, \ldots, 0}_{i_{n-k}-b \text{ times}}),$$

for some k between 1 and $n-1, 0 \leq i_j \leq k, i_1 + \cdots + i_{n-k} = k$. Here, position t is on one of the first b positions and $2^{i_{n-k}+1} - 1$ is on the $q(t)^{\text{th}}$ position, so that $i_{n-k} = n - q(t) + b$. Monomial $X_t^{\mathrm{mp}_\Gamma(X_t)+1} = X_t$ has weight 2^t and the corresponding monomial of Γ is $X_{q(t)}^{2^{n-q(t)+t}}$. We notice that this makes sense because $n - q(t) + t = i_{n-k} - (b - t) < i_{n-k}$ and thus $2^{n-q(t)+t} < \mathrm{mp}_\Gamma(X_{q(t)}) = 2^{i_{n-k}+1} - 1$. We conclude that in this case $q(t) = \max\{i \mid i \leq n, \mathrm{mp}_\Gamma(X_i) \neq 0\}$ and $r(t) = 2^{t+n-q(t)}$. ∎

Remark 1.44. The indices $q(t)$ of the previous lemma are such that we have $\mathrm{mp}_\Gamma(X_{q(t)}) \neq 0$. In other words, the corresponding position in the power vector of Γ is not zero.

In particular, let Γ have power vector of the form

$$(\underbrace{0, \ldots, 0}_{b \text{ times}}, 2^{i_1+1} - 1, \underbrace{0, \ldots, 0}_{i_1 \text{ times}}, \ldots, 2^{i_{n-k}+1} - 1, \underbrace{0, \ldots, 0}_{i_{n-k}-b \text{ times}}),$$

for some k between 1 and $n-1, 0 \leq i_j \leq k, i_1 + \cdots + i_{n-k} = k$.

If t is an integer for which X_t occurs in Γ, then there exists a positive integer l such that $t = i_1 + \cdots + i_{l-1} + l$ and $\mathrm{mp}_\Gamma(X_t) = 2^{i_l+1} - 1$. Thus, the corresponding integer $q(t)$ equals $i_1 + \cdots + i_{l-1} + i_l + l + 1$. This means that $X_{q(t)}$ is the next variable that occurs in Γ immediately after X_t. ♣

Corollary 1.45. *For Γ a H_n-graph, the set $T(\Gamma)$ equals $\{(\delta_{i,t}(\mathrm{mp}_\Gamma(X_t)+1) - \delta_{i,q(t)}r(t))_{i=1,\dots,n} \mid t \in \{1,\dots,n\}\}$. Here $\delta_{a,b}$ is the Kronecker's symbol, equal to 1 if $a=b$ and to 0 otherwise.* ∎

Definition-Notation 1.46. *Let G be a finite diagonal subgroup of $GL_n(\mathbb{C})$ and Γ a $G-$graph. We denote by $E(\Gamma)$ the set $\{v \in \mathrm{Jun}^+(G) | \langle v,u\rangle \geq 0, \forall u \in T(\Gamma)\}$. Here $T(\Gamma)$ is as in Definition 1.29, (5) and $\mathrm{Jun}^+(G)$ is the extended junior set of G, as in Definition 1.18.*

Example 1.47. For example, the set $E(\Gamma)$ for $n=4$ and Γ the H_4-graph $\{1, X_1, X_1, \dots, X_1^{14}\}$ is $\{\frac{1}{15}(1,2,4,8),\ e_2, e_3, e_4\}$. ♣

Lemma 1.48. *Let Γ be a H_n-graph with power vector $\mathrm{pv}(\Gamma) = (c_1,\dots,c_n)$. Then, the set $E(\Gamma)$ consists of the following elements:*

1. *e_t for those t such that $c_t = 0$ — that is if X_t doesn't occur in Γ;*
2. *$2^{(n-t+1) \pmod n} \star h_n$ for those t such that $c_t \neq 0$ – this is if X_t is in Γ.*

Proof:

To prove (1), let t be such that $c_t = 0$ and let u be a vector in $T(\Gamma)$. We want to prove that $\langle e_t, u\rangle$ is non-negative. By Corollary 1.45, u is of the form $(\delta_{i,s}(\mathrm{mp}_\Gamma(X_s)+1) - \delta_{i,q(s)}r(s))_{i=1,\dots,n}$, for some index s. Thus, $\langle e_t, u\rangle = \delta_{t,s}(\mathrm{mp}_\Gamma(X_s)+1) - \delta_{t,q(s)}r(s)$. Now, $c_t = 0$ and by Remark 1.44 we can not have $t = q(s)$, for any value of s. We deduce $\langle e_t, u\rangle = \delta_{t,s}(\mathrm{mp}_\Gamma(X_s)+1)$ which is a non-negative integer.

For (2), we consider an index t with $c_t \neq 0$. As before, we want to prove that for any u in $T(\Gamma)$ the scalar product $\langle 2^{(n-t+1) \pmod n} \star h_n, u\rangle$ is non-negative. Let u equal $(\delta_{i,s}(\mathrm{mp}_\Gamma(X_s)+1) - \delta_{i,q(s)}r(s))_{i=1,\dots,n}$, for some index s.

Then, the scalar product $\langle 2^{(n-t+1) \pmod n} \star h_n, u\rangle$ is:

$$\frac{1}{2^n-1}\sum_{i=1}^{n} 2^{(n-t+i) \pmod n} \times (\delta_{i,s}(\mathrm{mp}_\Gamma(X_s)+1) - \delta_{i,q(s)}r(s)) =$$

$$\frac{1}{2^n-1}[2^{(n-t+s) \pmod n}(\mathrm{mp}_\Gamma(X_s)+1) - 2^{(n-t+q(s)) \pmod n} r(s)].$$

So, it is enough to prove that the quantity $C(s,t) := 2^{(n-t+s) \pmod n} (\mathrm{mp}_\Gamma(X_s)+1) - 2^{(n-t+q(s)) \pmod n} r(s)$ is non-negative. We use Lemma 1.43 for a description of $q(s)$ and $r(s)$.

The first case is when X_s occurs in Γ. Then, $\mathrm{mp}_\Gamma(X_s) = 2^{l+1}-1$, for some integer l as in Lemma 1.32, $q(s) = (s+l+1) \pmod n$ and $r(s) = 1$. We get

$C(s,t) = 2^{(n+s-t) \pmod n} \times 2^{l+1} - 2^{(n+q(s)-t) \pmod n} = 2^{(n+s-t) \pmod n)+l+1} - 2^{(n+s-t+l+1) \pmod n}$, which is obviously a non-negative quantity.

We suppose now that X_s doesn't occur in Γ, this is $\mathrm{mp}_\Gamma(X_s) = 0$. As in Lemma 1.43, we take the set $\{i \mid 0 < i < s, \mathrm{mp}_\Gamma(X_i) \neq 0\}$. If this set is not empty, then $q(s) = \max\{i \mid 0 < i < s, \mathrm{mp}_\Gamma(X_i) \neq 0\}$ and $r(s) = 2^{s-q(s)}$. Then, we have $C(s,t) = 2^{(n-t+s) \pmod n} - 2^{(n-t+q(s)) \pmod n} \times 2^{s-q(s)}$. Now, if $s \leq t$, then, by definition of $q(s)$ we have also $q(s) \leq t$ and we deduce $(n-t+s) \pmod n = n+s-t$,$(n-t+q(s)) \pmod n = n-t+q(s)$ and $C(s,t) = 0$. If $s \geq t$, by definition of $q(s)$ we have $q(s) \geq t$ so $(n-t+s) \pmod n = s-t$, $(n-t+q(s)) \pmod n = q(s)-t$ and again $C(s,t) = 0$.

If X_s doesn't occur in Γ and the set $\{i \mid 0 < i < s, \mathrm{mp}_\Gamma(X_i) \neq 0\}$ is empty, we get $s \leq t$, $q(s) = \max\{i \mid i \leq n, \mathrm{mp}_\Gamma(X_i) \neq 0\}$ and $r(s) = 2^{n+s-q(s)}$. We therefore have $C(s,t) = 2^{(n-t+s) \pmod n} - 2^{(n-t+q(s)) \pmod n} \times 2^{n+s-q(s)}$. By definition of $q(s)$ we have $s < q(s)$ and also $q(s) \geq t$, so actually $s \leq t \leq q(s)$. We conclude that $(n-t+s) \pmod n = n-t+s$, $(n-t+q(s)) \pmod n = q(s)-t$ and that $C(s,t) = 2^{(n-t+s)} - 2^{q(s)-t+n+s-q(s)} = 0$.

We end the proof by showing that e_t, for $c_t = 0$ and $2^{(n-t+1) \pmod n} \star h_n$, for $c_t \neq 0$ are the only vectors that can occur in $E(\Gamma)$.

First, we remark that e_t, with $c_t \neq 0$ can not be in $E(\Gamma)$. This is because we can find an index s such that $t = q(s)$, as follows. By Remark 1.44, if there exists $i < t$ such that X_i occurs in Γ, then we put s the largest such i and by Lemma 1.43 we have $t = q(s)$. Then, $\langle e_t, (\delta_{i,s}(\mathrm{mp}_\Gamma(X_s)+1) - \delta_{i,q(s)} r(s))_{i=1,\ldots,n}\rangle$ is a negative integer equal to $-r(s)$. If for any $i < t$ the variable X_i doesn't occur in Γ and $t \neq n$, then for t with $\mathrm{mp}_\Gamma(X_t) > 1$ we put $s = t+1$ and for t with $\mathrm{mp}_\Gamma(X_t) = 1$ we put $s = \max\{i \mid i \leq n, \mathrm{mp}_\Gamma(X_i) \neq 0\}$. We have $t = q(s)$ and again, for $u := (\delta_{i,s}(\mathrm{mp}_\Gamma(X_s) + 1) - \delta_{i,q(s)} r(s))_{i=1,\ldots,n} \in T(\Gamma)$ we have $\langle e_t, u\rangle = -r(s) < 0$, so e_t can not be in $E(\Gamma)$. Now, if for any $i < t$ the variable X_i doesn't occur in Γ and $t = n$, the H_n−graph Γ is a linear one, only in X_t and we take s to be any index different of n. By Lemma 1.43, we have $t = q(s)$ and again for $u := (\delta_{i,s}(\mathrm{mp}_\Gamma(X_s)+1) - \delta_{i,q(s)} r(s))_{i=1,\ldots,n} \in T(\Gamma)$ we have a negative scalar product $\langle e_t, u\rangle = -r(s) < 0$. We conclude that if $c_t \neq 0$, then e_t can not be in $E(\Gamma)$.

By Lemma 1.42, the elements of $E(\Gamma)$ that are in $\mathrm{Jun}^+(H_n)$ are of the form $2^p \star h_n$, for some integer p between 1 and n. Let p be such that $2^p \star h_n$ is in $E(\Gamma)$. We want to prove that there exists t such that $c_t \neq 0$ and $p = (n-t+1) \pmod n$. First, if $p = 0$, then h_n is in $E(\Gamma)$ and we want to prove that t is one. We argue by contradiction and we suppose that $c_1 = 0$, that is $\mathrm{mp}_\Gamma(X_1) = 0$. Then, with notations as in Lemma 1.43, $q(1) = \max\{i \mid c_i \neq 0\}$ and $r(1) = 2^{n+1-q(1)}$. We would have, for $u = (\delta_{i,1} - \delta_{i,q(1)} r(1))_{i=1,\ldots,n} \in T(\Gamma)$, $\langle h_n, u\rangle = \frac{1}{2^n-1}(1 - 2^{q(1)-1} r(1)) = \frac{1}{2^n-1}(1-2^n)$ which is negative. This is a contradiction. So, for $p = 0$, we put $t = 1$ and we have $c_1 \neq 0$ and $p = (n-t+1) \pmod n$ as wanted.

We take now $p \neq 0$ and put $t = n - p + 1$. We prove that c_t is not zero. Again, we argue by contradiction and we suppose that $c_t = 0$, that is $\mathrm{mp}_\Gamma(X_t) = 0$. We take $u = (\delta_{i,t} - \delta_{i,q(t)} r(t))_{i=1,\dots,n}$, which is an element of $T(\Gamma)$. We have $\langle 2^p \star h_n, u\rangle = \frac{1}{2^n-1}(2^{(t-1+p) \pmod n} - 2^{(q(t)-1+p) \pmod n} r(t)) = \frac{1}{2^n-1}(1 - 2^{(q(t)-1+p) \pmod n} r(t))$. If we are in case 2 of Lemma 1.43, we have $q(t) < t$ and we deduce $n - q(t) + 1 > n - t + 1 = p$ and $n > q(t) + 1 - p$. We get $\langle 2^p \star h_n, u\rangle = \frac{1}{2^n-1}(1 - 2^{q(t)-1+p} r(t)) = \frac{1}{2^n-1}(1 - 2^{t-1+p}) = \frac{1}{2^n-1}(1 - 2^n)$ which is a contradiction. If we are in case 3 of Lemma 1.43, we have $q(t) > t$ and we deduce $n - q(t) + 1 < n - t + 1 = p$ and $n < q(t) - 1 + p$. We get $\langle 2^p \star h_n, u\rangle = \frac{1}{2^n-1}(1 - 2^{q(t)-1+p-n} r(t)) = \frac{1}{2^n-1}(1 - 2^{t-1+p}) = \frac{1}{2^n-1}(1 - 2^n)$ which is a contradiction. We conclude that c_t is not zero, as wanted. ■

Lemma 1.49. *Let Γ be a H_n–graph. The cone generated by $E(\Gamma)$, that is $\{\sum_{v \in E(\Gamma)} a_v v \mid a_v \geq 0, \forall v\}$, is the cone $\sigma(\Gamma)$ associated to Γ.*

Proof:

From the definition of the sets $\sigma(\Gamma)$, $E(\Gamma)$ and $T(\Gamma)$ it follows that $E(\Gamma)$ is contained in $\sigma(\Gamma)$. Let us prove now that the converse is true.

Without loss of generality, we can suppose that Γ has zero-skip. Thus the power vector of Γ, $\mathrm{pv}_\Gamma := (c_1, \dots, c_n)$ is of the form:

$$(2^{i_1+1} - 1, \underbrace{0, \dots, 0}_{i_1 \text{ times}}, \dots, \underbrace{2^{i_l+1} - 1}_{\text{place } (i_1+\cdots+i_{l-1}+l)}, \underbrace{0, \dots, 0}_{i_l \text{ times}}, \dots, 2^{i_{n-k}+1} - 1, \underbrace{0, \dots, 0}_{i_{n-k} \text{ times}}).$$

Let u be a vector of $\sigma(\Gamma)$. We want to prove that u is a linear combination with non-negative coefficients of vectors of $E(\Gamma)$. By Lemma 1.48, this is the same as to prove that the system:

$$\sum_{t \,:\, c_t = 0} x_t e_t + \sum_{t \,:\, c_t \neq 0} x_t (2^{(n-t+1) \pmod n} \star h_n) = u \tag{1.3.1}$$

with x_t as unknowns, admits a solution $(a_t), t \in \{1, \dots, n\}$, with $a_t \geq 0, \forall t$.

We look for all lines t of the previous system (1.3.1), with $c_t \neq 0$. We notice that such a line doesn't contain any of the unknowns x_s, with $c_s = 0$. Thus, we can isolate a subsystem of $n - k$ equations and $n - k$ unknowns, x_t corresponding to t with $c_t \neq 0$. Such an index t is of the form $i_1 + \cdots + i_{l-1} + l$, for l between 1 and $n - k$, where we put $i_0 = 0$. We therefore denote x_t by y_l, u_t – the corresponding constant term on the right hand-side, by v_l and we call y_l the variable x_t. The l^{th} equation of the new system is the equation corresponding to line $t = i_1 + \cdots + i_{l-1} + l$ in (1.3.1).

Let $A = (a_{l,j})_{l,j \in \{1,\dots,n-k\}}$ be the associated matrix in the $(n-k) \times (n-k)$–system and l and j two different indices between 1 and $n - k$. Then, we have:

$$a_{l,j} = \begin{cases} \frac{2^{l-j+i_j+\cdots+i_{l-1}}}{2^n-1} & \text{if l>j (under the main diagonal)} \\ \frac{1}{2^n-1} & \text{if l=j (on the main diagonal)} \\ \frac{2^{n+l-j-i_l-\cdots-i_{j-1}}}{2^n-1} & \text{if l<j (above the main diagonal)} \end{cases}$$

We denote by $b_{l,j} = a_{l,j}(2^n - 1)$ and by B the matrix $(b_{l,j})_{l,j\in\{1,\dots,n-k\}}$. We have $\det A = \frac{1}{(2^n-1)^{n-k}} \det B$. We compute $\det B$ by Gauss'rule: we keep the first line as a pivot and we "make zero" on the first column. Then, on line l, for $l \geq 2$ and column j, for $j \geq 2$ we get:

$$b_{l,j} - b_{l,1}b_{1,j} = \begin{cases} (1-2^n)b_{l,j} & \text{if } l \geq j \\ 0 & \text{else.} \end{cases}$$

Developing following the first column gives a $(n-k-1) \times (n-k-1)-$determinant, with $(1-2^n)$ on the main diagonal and zero above it:

$$\begin{vmatrix} (1-2^n) & 0 & 0 & \dots & 0 \\ (1-2^n)b_{3,2} & (1-2^n) & 0 & \dots & 0 \\ (1-2^n)b_{4,2} & (1-2^n)b_{4,3} & (1-2^n) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ (1-2^n)b_{n-k,2} & (1-2^n)b_{n-k,3} & (1-2^n)b_{n-k,4} & \dots & (1-2^n) \end{vmatrix}.$$

We conclude that $\det A$ is $\frac{(1-2^n)^{n-k-1}}{(2^n-1)^{n-k}} = \frac{(-1)^{n-k-1}}{2^n-1}$. This means that the system in the y_l's admits a solution. To compute it, we use Kramer's rule and calculate the determinants A_l where we replaced in A the l^{th} column by the column of constant terms v_l. Without loss of generality, we may suppose that $l \neq 1$; a similar computation holds for $l = 1$. As above, we associate to A_l a matrix $B_l = (2^n-1)A_l$. Thus $\det A_l = \frac{1}{(2^n-1)^{n-k}} \det B_l$ and it is enough to compute $\det B_l$. We do this also by Gauss rule: we keep the first line as a pivot and we "make zero" on the first column. After developing following the first column, we find:

$$\begin{vmatrix} 1-2^n & 0 & \dots & 0 & 0 & w_2 & 0 & \dots & 0 \\ c_{3,2} & 1-2^n & \dots & 0 & 0 & w_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{l-2,2} & c_{l-2,3} & \dots & 1-2^n & 0 & w_{l-2} & 0 & \dots & 0 \\ c_{l-1,2} & c_{l-1,3} & \dots & c_{l-1,l-2} & 1-2^n & w_{l-1} & 0 & \dots & 0 \\ c_{l,2} & c_{l,3} & \dots & c_{l,l-2} & c_{l,l-1} & w_l & 0 & \dots & 0 \\ c_{l+1,2} & c_{l+1,3} & \dots & c_{l+1,l-2} & c_{l+1,l-1} & w_{l+1} & 1-2^n & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n-k,2} & c_{n-k,3} & \dots & c_{n-k,l-2} & c_{n-k,l-1} & w_{n-k} & c_{n-k,l+1} & \dots & 1-2^n \end{vmatrix}$$

where $c_{i,j} = b_{i,j}(1-2^n)$ and $w_j = (v_j - b_{j,1}v_1)(2^n-1)$, with $b_{i,j}$ as before. Thus, we have a common factor $(1-2^n)^{n-k-2}(2^n-1) = (-1)^{n-k}(2^n-1)^{n-k-1}$. We can develop following the last $(n-k-1)-(l+1)+1 = n-k-l-1$ lines and get a new $(l-1)\times(l-1)-$determinant. To resume, up to now, we have $\det B_l = (-1)^{n-k}(2^n-1)^{n-k-1}\det C_l$, where:

$$\det C_l = \begin{vmatrix} 1 & 0 & \dots & 0 & 0 & v_2 - b_{2,1}v_1 \\ b_{3,2} & 1 & \dots & 0 & 0 & v_3 - b_{3,1}v_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{l-2,2} & b_{l-2,3} & \dots & 1 & 0 & v_{l-2} - b_{l-2,1}v_1 \\ b_{l-1,2} & b_{l-1,3} & \dots & b_{l-1,l-2} & 1 & v_{l-1} - b_{l-1,1}v_1 \\ b_{l,2} & b_{l,3} & \dots & b_{l,l-2} & b_{l,l-1} & v_l - b_{l,1}v_1 \end{vmatrix}$$

We develop $\det C_l$ following the last column:

$$\det C_l = (-1)^{1+(l-1)}(v_2 - b_{2,1}v_1) \begin{vmatrix} b_{3,2} & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ b_{l-2,2} & b_{l-2,3} & \dots & 1 & 0 \\ b_{l-1,2} & b_{l-1,3} & \dots & b_{l-1,l-2} & 1 \\ b_{l,2} & b_{l,3} & \dots & b_{l,l-2} & b_{l,l-1} \end{vmatrix} +$$

$$+(-1)^{2+(l-1)}(v_3 - b_{3,1}v_1) \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ b_{4,2} & b_{4,3} & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{l-2,2} & b_{l-2,3} & b_{l-2,4} & \dots & 1 & 0 \\ b_{l-1,2} & b_{l-1,3} & b_{l-1,4} & \dots & b_{l-1,l-2} & 1 \\ b_{l,2} & b_{l,3} & b_{l,4} & \dots & b_{l,l-2} & b_{l,l-1} \end{vmatrix} + \dots$$

$$\dots + (-1)^{(l-3)+(l-1)}(v_{l-2} - b_{l-2,1}v_1) \begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ b_{3,2} & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ b_{l-1,2} & b_{l-1,3} & \dots & b_{l-1,l-2} & 1 \\ b_{l,2} & b_{l,3} & \dots & b_{l,l-2} & b_{l,l-1} \end{vmatrix} +$$

$$+(-1)^{(l-2)+(l-1)}(v_{l-1} - b_{l-1,1}v_1) \begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ b_{3,2} & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ b_{l-2,2} & b_{l-2,3} & \dots & 1 & 0 \\ b_{l,2} & b_{l,3} & \dots & b_{l,l-2} & b_{l,l-1} \end{vmatrix} +$$

$$+(-1)^{(l-1)+(l-1)}(v_l - b_{l,1}v_1)\begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ b_{3,2} & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ b_{l-2,2} & b_{l-2,3} & \dots & 1 & 0 \\ b_{l-1,2} & b_{l-1,3} & \dots & b_{l-1,l-2} & 1 \end{vmatrix}.$$

We remark that, for an index $p, 1 \leq p \leq l-2$, we have $b_{l,l-1}b_{l-1,p} = 2^{-(l-1)+l+i_{l-1}} \times 2^{-p+(l-1)+i_p+\cdots+i_{l-2}} = b_{l,p}$. We deduce:

$$\frac{b_{l-1,2}}{b_{l,2}} = \frac{b_{l-2,3}}{b_{l,3}} = \cdots = \frac{b_{l-1,l-2}}{b_{l,l-2}} = \frac{1}{b_{l,l-1}}.$$

Thus, the first $l-3$ determinants in the above sum are zero and we obtain $\det C_l = -(v_{l-1} - b_{l-1,1}v_1)b_{l,l-1} + (v_l - b_{l,1}v_1) = -v_{l-1}b_{l,l-1} + v_l$. We conclude that $\det A_l = \frac{1}{(2^n-1)^{n-k}} \det B_l = (2^n-1)^{k-n} \times (-1)^{n-k}(2^n-1)^{n-k-1} \det C_l = \frac{(-1)^{n-k}}{(2^n-1)} \det C_l = \frac{(-1)^{n-k}}{(2^n-1)}(-v_{l-1}b_{l,l-1} + v_l)$.

So, for an index $l \neq 1$, we obtain $y_l = \det A_l / \det A = \frac{(-1)^{n-k}}{2^n-1} \times (-v_{l-1}b_{l,l-1} + v_l) \times \frac{2^n-1}{(-1)^{n-k-1}} = v_{l-1}b_{l,l-1} - v_l$. Let s be the index $i_1 + \cdots + i_{l-2} + l - 1$. By Lemma 1.32, this is an integer between 1 and n, with $c_s \neq 0$. By Lemma 1.43, part 1, we have $b_{l,l-1} = \mathrm{mp}_\Gamma(X_s) + 1$, $t = q(s)$ and $r(s) = 1$. By Corollary 1.45, we conclude that the solution a_t for the unknown x_t, with $c_t \neq 0$ and $t \neq 1$, is $a_t = \langle u, (\delta_{i,s}(\mathrm{mp}_\Gamma(X_s)+1) - \delta_{i,q(s)}r(s))_{i=1,\dots,n}\rangle$, where $(\delta_{i,s}(\mathrm{mp}_\Gamma(X_s)+1) - \delta_{i,q(s)}r(s))_{i=1,\dots,n}$ is a vector of $T(\Gamma)$, the vector associated to the ratio $X_s^{\mathrm{mp}_\Gamma(X_s)+1}/\mathrm{wt}_\Gamma(X_s^{\mathrm{mp}_\Gamma(X_s)+1})$. A similar computation gives $a_1 = \langle u, v(X_q^{\mathrm{mp}_\Gamma(X_q)+1}/\mathrm{wt}_\Gamma(X_q^{\mathrm{mp}_\Gamma(X_q)+1}))\rangle$, with $q = \max\{s \leq n \mid \mathrm{mp}_\Gamma(X_s) \neq 0\} = i_1 + \cdots + i_{n-k-1} + n - k$. By Definition 1.29, 1, the vector u is such that $\langle u, v(m/\mathrm{wt}_\Gamma(m))\rangle \geq 0$, for any monomial m in M – the set of all monomials in n variables. We conclude that $a_t \geq 0$, for all indices t with $c_t \neq 0$.

We compute now the solution a_t corresponding to t with $c_t = 0$. Such an index t is of the form $i_1 + \cdots + i_{l-1} + l + m$, for an index l between 1 and $n-k-1$ and an integer m with $1 \leq m \leq i_l$. We replace in line t of system 1.3.1 each a_s with $c_s \neq 0$ by its value. Such an index s with $c_s \neq 0$ is of the form $i_1 + \cdots + i_{j-1} + j$, for some j between 1 and $n-k-1$. We then have $a_t = u_t - \frac{1}{2^n-1} \sum\limits_{s\,:\,c_s\neq 0} a_s 2^{(n-s+t) \pmod n}$, where $a_s = v_{j-1}2^{i_{j-1}+1} - v_j, s \neq 1, j \neq 1$ and $a_1 = v_1 2^{i_1+1} - v_{i_1+\cdots+i_{n-k-1}+n-k}, s = i_1 + \cdots + i_{j-1} + j, j = 1$ The sum $\sum\limits_{s\,:\,c_s\neq 0} a_s 2^{(n-s+t) \pmod n}$ splits into two sums, following $j \leq l$ or $j > l$. We obtain $a_t = u_t - 2^m u_s$, for $s = i_1 + \cdots + i_{l-1} + l$. By Lemma 1.43, part 2 we have $q(t) = s$ and $m = r(t)$, so a_t, with $c_t = 0$, is nothing else but the scalar product $\langle u, v(X_t/\mathrm{mp}_\Gamma(X_t))\rangle$. By the same argument as before, we have that $a_t \geq 0$, if $c_t = 0$.

We conclude that the system 1.3.1 admits a unique solution $(a_t)_{t\in\{1,\ldots,n\}}$, with $a_t \geq 0$, for any index t. This proves the lemma. ■

Example 1.50. For a more explicit approach, let us actually see what happens for $n = 4$ and the H_4-graph $\Gamma = (1, X_1, X_1^2, \ldots, X_1^{14})$. The set $E(\Gamma)$ is given by the vectors e_2, e_3, e_4 and the vector associated to the matrix h_4 — this is $\frac{1}{15}(1, 2, 4, 8)$. Let $u = (u_1, u_2, u_3, u_4)$ be a vector of $\sigma(\Gamma)$. The solutions of the system (1.3.1) in this case are:

$$\begin{aligned} a_{v_4} &= \langle u, 15e_1\rangle, \\ a_{e_2} &= \langle u, e_2 - 2e_1\rangle, \\ a_{e_3} &= \langle u, e_3 - 4e_1\rangle, \\ a_{e_4} &= \langle u, e_4 - 8e_1\rangle. \end{aligned} \tag{1.3.2}$$

The monomial generators of $I(\Gamma)$ are X_1^{15}, X_2, X_3, X_4. The vectors associated to the ratios of those monomial generators are $15e_1, e_2-2e_1, e_3-4e_1$ and e_4-8e_1. By definition of $\sigma(\Gamma)$, the numbers $\langle u, 15e_1\rangle, \langle u, e_2-2e_1\rangle, \langle u, e_3-4e_1\rangle$ and $\langle u, e_4 - 8e_1\rangle$ are non-negative. Thus the solutions given by (1.3.2) are also non-negative numbers, as wanted. ♣

Corollary 1.51. *For any H_n-graph Γ, the elements of the set $E(\Gamma)$ are linearly independent.*

Proof:

We take a null linear combination of the vectors v of $E(\Gamma)$, this is:

$$\sum_{v\in E(\Gamma)} a_v v = 0.$$

By Lemma 1.49, for any vector u the solution of the linear system

$$\sum_{v\in E(\Gamma)} x_v v = u \tag{1.3.3}$$

is given by $\langle u, w(v)\rangle$, where $w(v)$ is as follows. If v is e_t, then $w(v)$ is the vector associated to the ratio $X_t/\mathrm{mp}_\Gamma(X_t)$. Else, this is if $v \in E(\Gamma)$ is different from a vector of the basis, the coordinate corresponding to v in the power vector of Γ is a positive integer $c_t \neq 0$. If there exists an index $j < t$ such that $c_j \neq 0$ put $s = \max\{j < t | c_j \neq 0\}$, else put $s = \max\{j \neq n | c_j \neq 0\}$. Then, $w(v)$ is the vector associated to the ratio $X_s^{\mathrm{mp}_\Gamma(X_s)+1}/\mathrm{wt}_\Gamma(X_s^{\mathrm{mp}_\Gamma(X_s)+1})$. It is now clear that if u is the null vector, the system (1.3.3) has zero solution. ■

Corollary 1.52. *For any H_n-graph Γ, the cone $\sigma(\Gamma)$ is $n-$dimensional.* ■

Lemma 1.53. *Let* Γ *be an* H_n*−graph. Then, any vector* $2^k \star h_n, k \geq 0$, *is a linear combination with integer coefficients of the vectors of the set* $E(\Gamma)$.

Proof:

We consider the linear system (1.3.1) for the particular choice $u = 2^k \star h_n$, for some non-negative integer k. Following the proof of Lemma 1.49, the solution $(a_1, \ldots, a_n)$ of such a system can be described as follows.

For an index t such that $c_t \neq 0$, there exists an positive integer l such that $t = i_1 + \cdots + i_{l-1} + l$. We put $s = i_1 + \cdots + i_{l-2} + l - 1$. Remark that if $t = 1$, we have to consider some other index s, but this case can be treated similarly. We then have that $a_t = u_s b_{l,l-1} - u_t$, this is $a_t = \frac{1}{2^n - 1}(2^{(s+k-1) \pmod n} b_{l,l-1} - 2^{(t+k-1) \pmod n})$, where $b_{l,l-1} = 2^{i_{l-1}+1}$. If $s + k - 1$ and $t + k - 1$ are both greater than n or both less than n, this gives $a_t = 0$. Otherwise, that is if $s \leq n - k + 1 \leq t$, we deduce $a_t = 2^{t-n+k-1}$.

Now, if the index t is such that $c_t = 0$, we write t as $i_1 + \cdots + i_{l-1} + l + m$, for $1 \leq m \leq i_l$. Let us put $s = i_1 + \cdots + i_{l-1} + l$. Then, a_t is given by $u_t - 2^m u_s$. We replace u_t and u_s by their values and we see that again two cases can occur. If $s + k - 1$ and $t + k - 1$ are both greater than n or both less than n, this gives $a_t = 0$. Otherwise, that is if $s \leq n - k + 1 \leq t$, we deduce $a_t = -2^{t-n+k-1}$. ■

Remark 1.54. An accurate analysis of the above proof, shows that actually, if $2^k \star h_n$ doesn't occur in $E(\Gamma)$, then it is a combination with at least one negative coefficient of the vectors of the set $E(\Gamma)$. ♣

Corollary 1.55. *For any* H_n*−graph* Γ, *the set* $E(\Gamma)$ *is a basis of the lattice* $N := \mathbb{Z}^n + h_n\mathbb{Z}$.

Proof:

The relation $e_1 = (2^n - 1)h_n - 2e_2 - \cdots - 2^{n-1}e_n$ shows that a basis of the lattice N is $\{h_n, e_2, \ldots, e_n\}$. So, it is enough to show that each of those vectors is a linear combination with integer coefficients of the elements of $E(\Gamma)$. For the vector h_n this is a consequence of the previous lemma.

For a vector $e_i, 2 \leq i \leq n$, we use again the proof of Lemma 1.49 that gives the solution of (1.3.1), for $u = e_i$. We write e_i as a linear combination with coefficients $a_t, 1 \leq t, \leq n,$, of the vectors of $E(\Gamma)$. Then, for any integer $t, 1 \leq t \leq n$, the coefficient a_t is the scalar product $\langle e_i, v(p_t/\mathrm{wt}_\Gamma(p_t))\rangle$, for some monomial p_t. The vector $v(p_t/\mathrm{wt}_\Gamma(p_t))$ has integer coefficients because is the vector associated to the fraction $p_t/\mathrm{wt}_\Gamma(p_t)$. So, a_t is also an integer, the power of the i^{th} variable in the fraction $p_t/\mathrm{wt}_\Gamma(p_t)$.

■

Corollary 1.56. *$X(\mathrm{Fan}(H_n))$ is a smooth variety.*

Proof:

By Definition 1.29, the variety $X(\mathrm{Fan}(H_n))$ is the toric variety associated to the fan $\mathrm{Fan}(H_n)$. This variety is obtained by gluing the affine pieces $\mathrm{Spec}(\mathbb{C}[\sigma^\vee(\Gamma) \cap N^\vee])$, where N is as in 1.1.1 and $\bullet^\vee$ denotes the dual, for a lattice as well as for a cone. It is enough to see that every such affine piece is a copy of $\mathbb{C}^n$. For this, we use the "smoothness criterion" of [34], Theorem 1.10, page 10. According to this, it is enough to prove that every $E(\Gamma)$ is part of a basis for the lattice $N := \mathbb{Z}^n + h_n\mathbb{Z}$. This follows from Corollary 1.55. ∎

Lemma 1.57. *For any H_n−graph Γ, we have $S(\Gamma) = \sigma^\vee(\Gamma) \cap N^\vee$.*

Proof:

Let $(c_1, \ldots, c_n)$ be the power vector of the H_n−graph Γ.

By definitions of $\sigma(\Gamma)$ and $S(\Gamma)$, the inclusion $S(\Gamma) \subset \sigma^\vee(\Gamma) \cap N^\vee$ follows immediately. Now, for the reverse inclusion, it is enough to prove that $\sigma^\vee(\Gamma) \cap N^\vee$ is generated by a set of vectors contained in $S(\Gamma)$. The set $\sigma^\vee(\Gamma) \cap N^\vee$ is the set $\{v = (v_1, \ldots, v_n) \in \mathbb{Z}^n \mid \langle u, v\rangle \geq 0, \forall u \in \sigma(\Gamma); \langle m, v\rangle \in \mathbb{Z}, \forall m \in N\}$. We write any vector $v = (v_1, \ldots, v_n)$ of $\sigma^\vee(\Gamma) \cap N^\vee$ as a linear combination with non-negative integers of some vectors of the form $v(p/\mathrm{wp}_\Gamma(p))$, where p runs in the set of all monomials. We claim that any vector v in $\sigma^\vee(\Gamma) \cap N^\vee$ is a linear combination with non-negative integer coefficients of the vectors of $T(\Gamma)$. Now, the set $T(\Gamma)$ is a subset of $S(\Gamma)$, which ends the proof.

First, we make some considerations on the set $\sigma^\vee(\Gamma) \cap N^\vee$. Let v be an element in $\sigma^\vee(\Gamma) \cap N^\vee$. By Lemma 1.49, any cone $\sigma(\Gamma)$ is generated by the corresponding set $E(\Gamma)$ of Definition 1.46. So, it is enough to ask that $\langle u, v\rangle \geq 0, \forall u \in E(\Gamma)$. By (1.1.1) and definition of a dual lattice, it is also enough to ask that $\langle e_i, v\rangle \in \mathbb{Z}$, for all indices i, and $\langle v(g), v\rangle \in \mathbb{Z}$, for all g of H_n (with Notation 1.4). The group H_n is cyclic, generated by the matrix g_n (see notations at the beginning of the section). Each element g of H_n is of the form $g_n^{2^i}$, for some integer i between 0 and $n-1$. Thus, the vector $v(g) = v(g_n^{2^i})$ is nothing else but the vector $2^{(n-i+1) \pmod n} \star h_n$. We deduce that, for any integer t, the scalar product $\langle 2^{(n-t+1) \pmod n} \star h_n, v\rangle$ is an integer. Combining with Lemma 1.48, we deduce that for t with $c_t \neq 0$, the scalar product $\langle 2^{(n-t+1) \pmod n} \star h_n, v\rangle$ is a non-negative integer and for any t with $c_t = 0$, the integer $\langle e_t, v\rangle$ is also non-negative.

Now, we consider the following system, with x_i as unknowns:

$$v = \sum_{i=1}^{n} x_i \cdot v(X_i^{\mathrm{mp}_\Gamma(X_i)+1}/\mathrm{wt}_\Gamma(X_i^{\mathrm{mp}_\Gamma(X_i)+1}).$$

We want to prove it has a non-negative integer solution. The matrix of this system is formed with $(i_l + 1) \times (i_l + 1)$−matrices of the form:

$$\begin{pmatrix} 2^{i_l+1} & -2 & -2^2 & \dots & -2^{i_l-1} & -2^{i_l} \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

To compute such a determinant it is enough to compute the determinant of each block. For this, we develop following the first column. To compute the solution of the system, we use again Kramer's rule. Let us denote it by $(a_t)_{t\in\{1,\dots,n\}}$. We have $a_t = \langle 2^{(n-t+1) \pmod n} \star h_n, v\rangle$, if t is such that $c_t \neq 0$ and $a_t = \langle e_t, v\rangle$, if t is such that $c_t = 0$. As shown at the beginning of this lemma, each a_t is a non-negative integer, which ends the proof. ■

Lemma 1.58. *The Euler number of the toric variety $X(\mathrm{Fan}(H_n))$ is 2^n-1.*

Proof:

We follow Section 1.1.1 to see that the Euler number of $X(\mathrm{Fan}(H_n))$ is the number of $n-$dimensional cones of $\mathrm{Fan}(H_n)$, denoted by $\mathrm{Fan}(H_n)(n)$. By Definition 1.29, $\mathrm{Fan}(H_n)(n)$ is nothing else but the set $\{\sigma(\Gamma)|\Gamma \in \mathrm{Graph}(H_n)\}$. Now, it is clear that two different H_n-graphs give two different cones in $\mathrm{Fan}(H_n)$. Thus, we have $\#\mathrm{Fan}(H_n)(n) = \#\mathrm{Graph}(H_n)$. Corollary 1.36 then gives $\#\mathrm{Graph}(H_n) = 2^n - 1$. ■

Theorem 1.1 For any non-negative integer n, the quotient $\mathbb{A}^n/\mu_{2^n-1}$ admits a smooth crepant resolution of singularities which is the $\mu_{2^n-1}-$Hilbert scheme of $\mathbb{A}^n$.

Proof:

By Theorem 1.38, $\mu_{2^n-1}-\mathrm{Hilb}\mathbb{A}^n$ is the variety $V(\mathrm{Graph}(H_n))$ obtained by gluing together all the $\mathrm{Spec}\mathbb{C}[S(\Gamma)]$, when Γ runs over $\mathrm{Graph}(H_n)$. By Lemma 1.57, this is the same as the toric variety $X(\mathrm{Fan}(H_n))$. We apply Corollary 1.56 to conclude that $\mu_{2^n-1}-\mathrm{Hilb}\mathbb{A}^n$ is smooth. In particular, the μ_{2^n-1}-Hilbert scheme of $\mathbb{A}^n$ provides a toric resolution of the quotient $\mathbb{A}^n/\mu_{2^n-1} = \mathbb{A}^n/H_n$, by subdivision of the cone Δ_n into the sub-cones $\sigma(\Gamma), \Gamma \in \mathrm{Graph}(H_n)$.

Now, for crepancy, we use the equivalence stated in [8], page 656. Following this, two of the necessary conditions are satisfied: by Lemma 1.58, the Euler number of $\mu_{2^n-1}-\mathrm{Hilb}\mathbb{A}^n$ is the cardinal of μ_{2^n-1} and by the above arguments $\mu_{2^n-1}-\mathrm{Hilb}\mathbb{A}^n$ is smooth. This gives the crepancy. ■

Remark 1.59. Another proof of the crepancy can be given. For example, one can use Reid's result (cf. [38] or [9], Theorem 1.13) stating that the crepancy follows because the exceptional divisors are given by primitive vectors (this is vectors in the junior simplex). ♣

1.4 Miscellaneous remarks

Remark 1.60. About the exceptional set. The $\mu_{2^n-1}-$Hilbert scheme of $\mathbb{A}^n$ is obtained by adding new rays in the cone σ_0. Each ray corresponds to an element of $\mathrm{Jun}\mu_{2^n-1} = \mathrm{Jun}(H_n)$. Thus, the exceptional divisors are in one-to-one correspondence with the junior elements associated to μ_{2^n-1}. We therefore denote such a divisor by $D(h)$, for some element h of μ_{2^n-1} with $\mathrm{age}(v(h)) = 1$. Here, $v(h)$ is the vector associated to an element h of μ_{2^n-1}. We want to see when the intersection of k exceptional divisors $D(h_1), \ldots, D(h_k)$, gives a $(n-k)-$dimensional variety. Such a $(n-k)-$dimensional variety is given by a $k-$dimensional cone of the fan $\mathrm{Fan}(\mu_{2^n-1})$. This means actually that we want the rays $v(h_1), \ldots, v(h_k)$ to define a $k-$dimensional cone. The fan $\mathrm{Fan}(\mu_{2^n-1})$ is defined by help of $n-$dimensional cones. So, a $k-$dimensional cone containing $v(h_1), \ldots, v(h_k)$ is the intersection of k copies of $n-$dimensional cones, each including one of the rays $v(h_i)$ and such that their intersection contains all the rays $v(h_i), i \in \{1, \ldots, k\}$.

We translate this in terms of $\mu_{2^n-1}-$graphs. Let $\mathrm{Graph}(\mu_{2^n-1}, h)$ denote the set of all $\mu_{2^n-1}-$graphs containing $v(h)$ as a ray. The above condition means that the intersection $\cap_{i=1}^{k}\mathrm{Graph}(\mu_{2^n-1}, h_i)$ is not empty. ♣

Remark 1.61. Some counterexamples

The above method – of subdividing the cone σ_0 – fails to give a smooth crepant resolution of singularities for other groups. For example, we consider the action of the cyclic group μ_{40} of order 40 acting by weights $1, 3, 9$ and 27 respectively, on $\mathbb{A}^4$. The quotient $\mathbb{A}^4/\mu_{40}$ is also a Gorenstein quotient singularity, with isolated singularity at the origin. The variety $\mu_{40}-\mathrm{Hilb}\mathbb{A}^4$ fails to be a crepant resolution of $\mathbb{A}^4/\mu_{40}$. This is mainly because the number of $\mu_{40}-$graphs in this case is not equal to the cardinal of the group. Thus, Lemma 1.58 doesn't stand anymore. We remark also that in this case even if we deform along a principal direction, the result might be a set which is not a $\mu_{40}-$graph. If we adopt the convention to represent a monomial by a point with an associated weight (marked by a number), then the $\mu_{40}-$graph of Figure 1.5 deformed along the principal direction X_1 (with ratio X_1/X_2) gives a set that is not a $\mu_{40}-$graph. This is because condition 2 of Definition 1.21 is not fulfilled.

♣

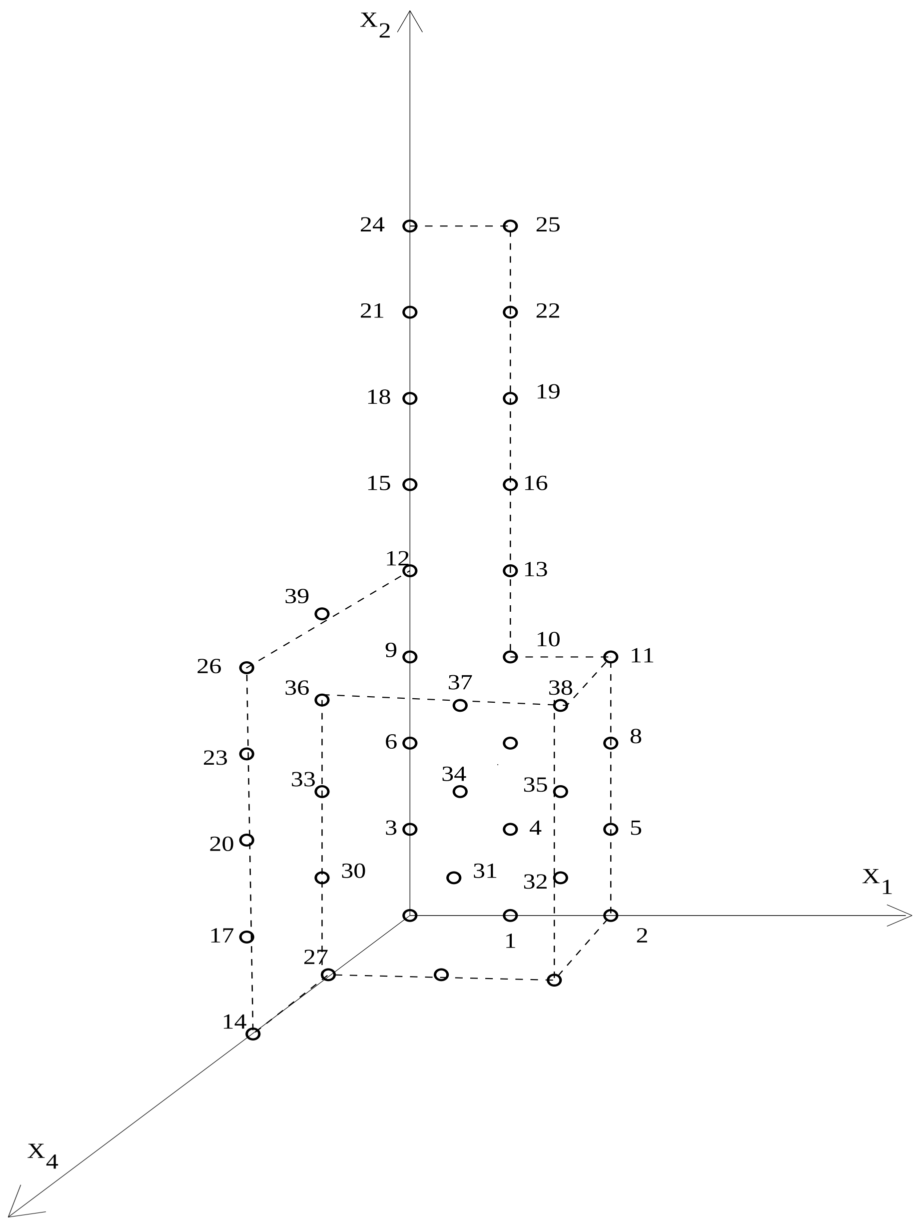

Figure 1.5: A $\mu_{40}-$graph providing no $\mu_{40}-$graph by principal deformation.

1.4.1 Relation with the McKay correspondence

According to Reid, one can ask the following (known also as the Classical McKay correspondence):

Question 1.62. *(strong McKay correspondence) Let G be a finite subgroup of $SL_n(\mathbb{C})$ acting on $\mathbb{A}^n$, such that the quotient variety $\mathbb{A}^n/G$ has only iso-*

lated singularities. We suppose that $\mathbb{A}^n/G$ admits a crepant resolution of singularities $f : X \to \mathbb{A}^n/G$. Then, the non-zero Betti numbers of X are:

$$\dim_{\mathbb{C}} H^{2j}(X, \mathbb{C}) = \#\{ \textit{ age } \ j \textit{ conjugacy classes of } G\}, \forall j \in \{1, \ldots, n-1\}.$$

In particular, the topological Euler number $e(G)$equals the number of conjugacy classes of G.

This is a natural generalization of the classical McKay correspondence, saying that, for $n = 2$, with the same notations, there is a bijection between the irreducible representations of G and a basis for he cohomology $H^*(X, \mathbb{C})$. By Batyrev, Craw, Dais, Reid et al (see for example [9], Theorem 2.39), it is known that Question 1.62 has a positive answer.

Thus also for the group H_n, the previous question has a positive answer. The existence of a crepant resolution follows from Theorem 1.1 and we have $X = \mu_{2^n-1}-\mathrm{Hilb}\mathbb{A}^n$. The relation between age conjugacy classes of μ_{2^n-1} and the cohomology of X conjectured before holds. For example, in the four-dimensional case, there is one element of age zero, corresponding to H^0. Following Lemma 1.41 and Corollary 1.42, we find four elements of age 1 – respectively $h_4, 2 \star h_4, 2^2 \star h_4, 2^3 \star h_4$, two elements of age 2 – which are $t \star h_4$ for t in the set $\{3, 5, 6, 9, 10, 12\}$ and again – as expected – four elements of age four – given by $t \star h_4$ for t in $\{7, 11, 13, 14\}$. Thus, the McKay conjecture holds in the form $1 + 4 + 6 + 4 = 15$.

In this way, the geometry of crepant resolution of the $\mu_{2^n-1}-$Hilbert scheme of $\mathbb{A}^n$ and the representation theory of the group μ_{2^n-1} are related. This makes a link with the following conjecture (cf. [36], Principle 1.1).

Conjecture 1.63. (Geometrical McKay correspondence) *Let V be a variety and G a subgroup of the group of automorphism of V. We suppose that the quotient variety V/G has a crepant resolution $f : Y \to V/G$. Then, the derived category of coherent sheaves on Y and the derived category of $G-$equivariant sheaves on V are equivalent by help of a Fourier-Mukai transform.*

To end the section, we follow [30] and [19] to show how to relate the representation theory of μ_{2^n-1} to the theory of graphs. We recall the following construction.

Construction 1.64. Let G be a finite group and $\mathrm{Irr}(G) = \{\rho_0, \rho_1, \ldots, \rho_l\}$ the set of its irreducible representations, where ρ_0 is the trivial representation. We suppose that G is a finite subgroup of $GL_n(\mathbb{C})$ and that R is the $n-$dimensional representation giving the inclusion $G \hookrightarrow GL_n(\mathbb{C})$. For each index t among $1, \ldots, l$ we consider the representation $R \otimes \rho_t$ and write it as a sum of irreducible representations. We associate to $\mathrm{Irr}(G)$ the following quiver:

- vertices of the quiver are integers $0, 1, \ldots, l$;
- for a fixed index t, if ρ_s occurs in the decomposition of $R \otimes \rho_t$, put an arrow from t to s. This quiver is called the McKay quiver.

See Figure 1.64 for a picture of the McKay quiver in the case of the group μ_7 seen as a subgroup of $SL_3(\mathbb{C})$ by help of H_3. ♣

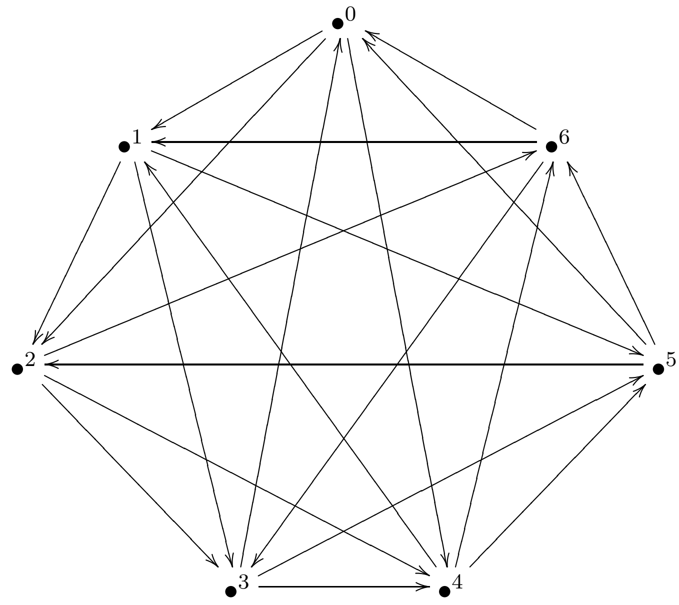

Figure 1.6: McKay quiver for H_3.

1.4.2 $G-$graphs and algorithms for crepant resolutions

In [10], one can find a generalization of the Hirzebruch-Jung algorithm (see Example 1.5) to the three-dimensional case of toric quotient varieties $\mathbb{A}^3/A$, with A a finite abelian group, subgroup of $SL_3(\mathbb{C})$. It is well known by the Shephard-Todd-Chevalley theorem that for a group Q generated by pseudoreflections, this is matrices T such that $\text{rank}(T - I_n) = 1$, acting on $\mathbb{A}^n$, there is an isomorphism $\mathbb{A}^n/Q \simeq \mathbb{A}^n$. Therefore, the group A can be supposed to have no pseudoreflections. We apply this algorithm for the group H_3 (which has no pseudoreflections). Combining with Nakamura's method of $G-$graphs, this algorithm can be generalized for $H_n, n > 3$.

First, we introduce some definitions and notations. As in the previous sections, $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

Definition 1.65. *(cf. [9], 3.1: "regular triangle") Let v_1, v_2, v_3 be three vectors in Δ_3, seen as points of the plane $\{(a, b, c) | a + b + c = 1\}$ and denote by s_1, s_2, s_3 the vectors $\overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, \overrightarrow{v_3v_1}$. We say that v_1, v_2, v_3 form a regular triangle if*

1. *any two of the vectors s_1, s_2, s_3 form a basis of $\mathbb{Z}^2$,*

2. *there is a relation of the form $\pm s_1 \pm s_2 \pm s_3 = 0$.*

We say that Δ_3 has a regular tessellation if it can be subdivided into a finite number of regular triangles.

The idea of the algorithm Craw-Reid is that one can reduce the construction to the two-dimensional case, that is to the Hirzebruch-Jung algorithm. A method to solve toric singularities is to subdivide the corresponding fan until one gets a non-singular variety. We consider the quotient $\mathbb{A}^3/A$, given by the lattice $\mathbb{Z}^3 + \sum_{g\in A} v(g)\mathbb{Z}$ – as in Equation 1.1.1 – and fan the cone σ_0. To subdivide the cone σ_0 by adding extra-rays is the same as to subdivide the triangle Δ_3 by adding junior points. Following this idea, the **Craw-Reid algorithm** provides a method to recover $A\text{−Hilb}\mathbb{A}^3$ by regular tessellation of Δ_3. The main steps are the following:

• Consider the set of junior points of the group A, this is the set $\mathrm{Jun}(A) = \{v(a) \in \Delta_3 | a \in A\}$.

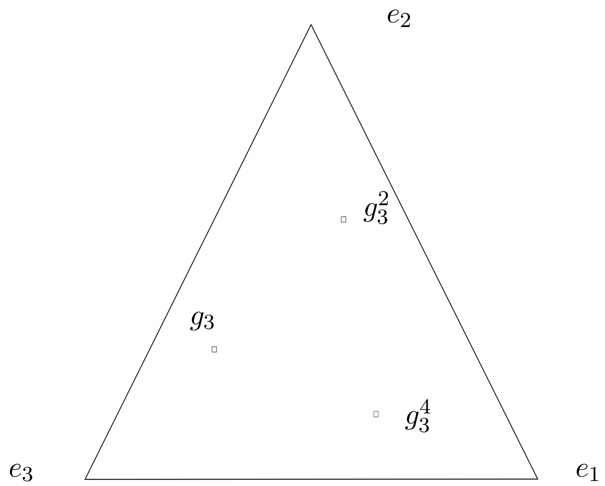

Figure 1.7: Junior points for $1/7(1, 2, 4)$

• Take each e_i (seen as a point in Δ_3) in turn as origin and consider $\overrightarrow{e_ie_{i-1}}, \overrightarrow{e_ie_{i+1}}$ as coordinate axes (here the indices are cyclic). Construct the Newton polygon of the junior points of $\Delta_3 \setminus \{e_i\}$ (as in the case $n = 2$). One can also see this like follows. Suppose A is a group of the form $1/r(a_1, \dots, a_n)$ as in Notation 1.4. Erase the i^{th} coordinate of A and get a new group A_i. It is also a cyclic group. Let it act on the affine plane $\mathbb{A}^2$. By a suitable choice of primitive root of unity, one can suppose that A_i acts by weights 1 and w_i on $\mathbb{A}^2$. We apply the Hirzebruch-Jung continued fractions algorithm for $\mathbb{A}^2/A_i$. The resulting Newton polygon – transposed in the coordinate system with axes $\overrightarrow{e_ie_{i-1}}, \overrightarrow{e_ie_{i+1}}$ – is nothing else but the Newton polygon of the junior points of $\Delta_3 \setminus \{e_i\}$.

• Let P be a junior point of A. Fix again an "origin" e_i and let $Q_1, \dots, Q_t$ be all the junior points of the Newton polygon draw before in the coordinate system with axes $\overrightarrow{e_i e_{i-1}}, \overrightarrow{e_i e_{i+1}}$. Take Q_k one of those points. We can associate to the line $Q_k e_i$ an integer, called its "power": this is the integer on the corresponding position k of the Hirzbruch-Jung continued fraction for $w_i/\#A_i$. See also the Example 1.66 bellow for an explicit computation.

Subdivide Δ_3 into regular triangles as follows. Every time two (or more) lines meet in a junior point, the line with bigger power is prolongated, but its power will diminish by the number of lines it defeats. Like this, a line of power two is not prolongated and two (or more) lines with same powers meeting in a common point are not prolongated either. See [9] for more details.

Example 1.66. We consider here the case of the group H_3. Fix ε a 7^{th} primitive root of the unity and put g_3 the diagonal matrix $\operatorname{diag}(\varepsilon, \varepsilon^2, \varepsilon^4)$. Identify this matrix with the vector $\frac{1}{7}(1,2,4)$ and denote it – by abuse – also by g_3.

• The junior points are the one corresponding to g_3, g_3^2 and g_3^4, this is $1/7(1,2,4), 1/7(2,4,1)$ and respectively $1/7(4,1,2)$ (see Figure 1.7).

• We take e_1 to be the origin and $\overrightarrow{e_1 e_3}$ and $\overrightarrow{e_1 e_2}$ coordinate axes. We consider the group A_1 of order 7 acting by weights 2 and 4 on the affine plane.

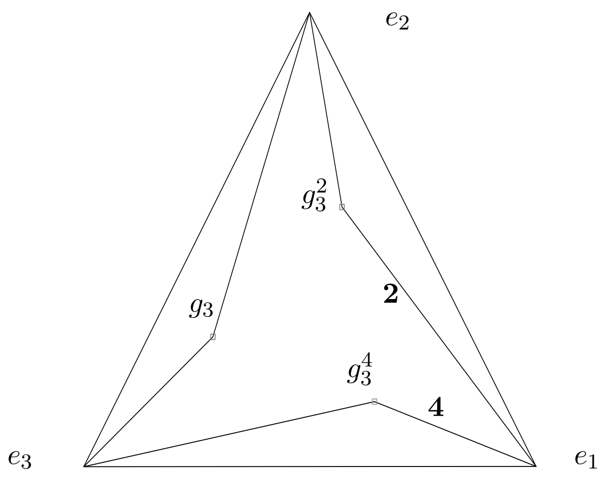

Figure 1.8: Newton polygons for $1/7(1,2,4)$

By a suitable choice of primitive root, this is the same as the action by weights 1 and 2 so one can apply the Hirzbruch-Jung algorithm for the group $1/7(1,2)$. We get the continuous fraction $[4,2]$. For the subdivision of the junior simplex Δ_3, this means that there are two lines going out of e_1 :

one is $e_1g_3^2$ with power 2 an the other is $e_1g_3^4$ with power 4. For e_2 and e_3, we recover actually the same action. This is not the case in general, but it is due to the particular action of H_3 on $\mathbb{A}^3$. The result is shown in Figure 1.8: it is a triangle with lines with "powers" going out of each corner.

• Now subdivide Δ_3 into regular triangles. For example, the line $e_1g_3^2$ has power 4 so it is prolongated up to g_3; it meets two other lines $e_3g_3^2$ and the prolongation of the line $e_2g_3^4$. Thus line $e_1g_3^2$ "defeats" two other lines, so once it arrives in g_3 it doesn't extend anymore (because its power is now two). The result is shown in Figure 1.9. ♣

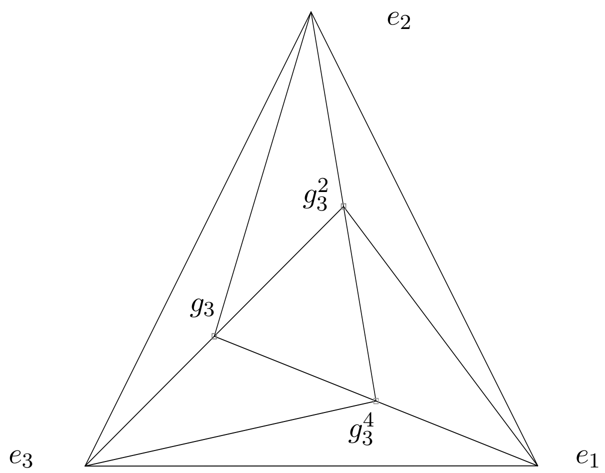

Figure 1.9: Regular triangles for $1/7(1,2,4)$

Remark 1.67. The cones obtained by the regular tessellation of the Craw-Reid algorithm are the ones obtained by help of Graph(G). To see this, it is enough to pass to the dual lattice M of all monomials. As shown in Section 3.5 of [9], for a given group A, for each line in the corresponding regular tessellation of Δ_3 we get a fraction $m : m'$, with m and m' two monomials in M. This is nothing else but the ratio introduced in Definition 1.24 for passing from a cone $\sigma(\Gamma)$ to a cone $\sigma(\Gamma')$. Thus, to recover $A-\mathrm{Hilb}\mathbb{A}^3$ the two methods are equivalent: the Craw-Reid algorithm and Nakamura's computation of Graph(A). The advantage of the first method is that it provides a geometric way to obtain $A-\mathrm{Hilb}\mathbb{A}^3$: it is enough to make a good subdivision of the junior simplex by help of junior points (see also the following Remark). This method gives also a link with the McKay correspondence, as shown in Section 1.4.1. The disadvantage of the Craw-Reid algorithm is a certain ambiguity of how to make the regular tessellation, as explained in [9], Section 3.4. This is compensated by Nakamura's method, which gives an infallible computational algorithm. ♣

Remark 1.68. Craw and Reid give a very nice geometrical interpretation of a $G-$graph, called tessellation of the plane. We show how this works for the example of the group H_3.

In general, we remark that, for a group G subgroup of $SL_3(\mathbb{C})$, the monomial $X_1X_2X_3$ is invariant by the group action on $\mathbb{A}^3$, this is on $\mathbb{C}[X_1, X_2, X_3]$. Thus, the monomial $X_1X_2X_3$ is contained in each ideal defining a $G-$cluster at the origin. So, a $G-$graph contains monomials in only two of the three variables X_1, X_2, X_3.

This allows to associate to a $G-$cluster Γ, a $2-$dimensional diagram $\mathcal{D}_\Gamma$, having "axes" at $120°$ (X_1 horizontal, X_2 at $120°$ clock-wise, X_3 at $120°$ anti-clock-wise). In this diagram, each monomial is represented by a hexagon. For example, for the H_3-graph $\Gamma_1 := \{1, X_1,\ X_1^2,\ X_1^3,\ X_3, X_1X_3, X_1^2X_3\}$, with power vector $(3, 0, 1)$, the corresponding diagram is represented in Figure 1.10.

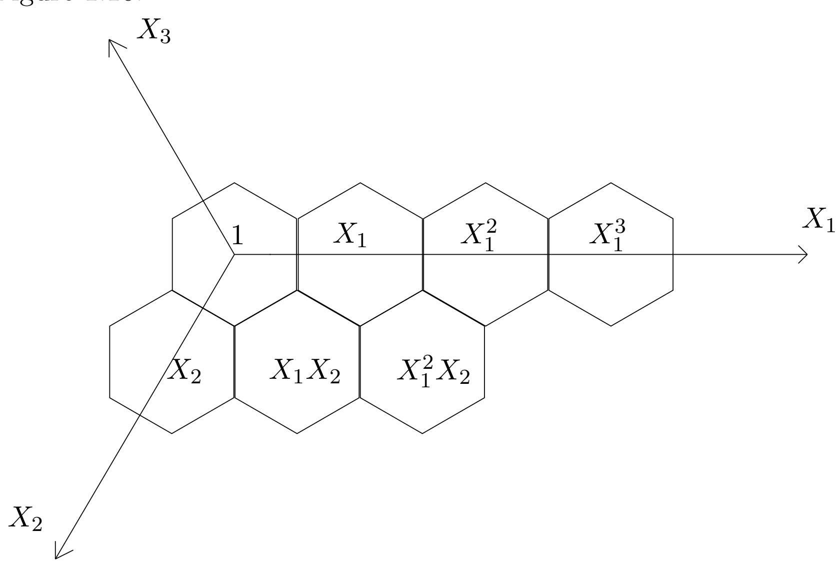

Figure 1.10: Diagram $\mathcal{D}_{\Gamma_1}$ for the group H_3.

Now, let m_0 be a monomial in the $G-$graph Γ and m another monomial (in two of the three variables) with the same associated character as m_0 (see Definition 1.20). This means that m/m_0 is a $G-$invariant fraction and that $\mathrm{wt}_\Gamma(m) = m_0$. We move the diagram $\mathcal{D}_\Gamma$ by parallel transport, mapping the hexagon corresponding to m_0 into the one corresponding to m; we denote this by $m_0 \rightrightarrows m$. We remark that the parallel transport respects representations: if p_0 is another monomial in the same diagram $\mathcal{D}_\Gamma$ as m_0, such that p_0 is mapped into p by parallel transport $m_0 \rightrightarrows m$, the fractions m_0/m and p_0/p correspond to the same character. In particular, suppose that the hexagons corresponding to some monomials p_0 and p_0' are neighboring in the initial diagram $\mathcal{D}_\Gamma$ – which means that p_0' is obtained from

p_0 by multiplication with X_1, X_2 or X_3. Then, in the transported diagram, the corresponding hexagons p and p' are still neighboring. The collection of all the transported diagrams $\mathcal{D}$ tessellate the plane, meaning that each monomial (in two of the three variables) belongs to a unique [transported] diagram. In other words, all those diagrams fill-up the plane by translations. For example, in the case of the graph H_3 before, the result is shown in Figure 1.11. The monomial 1 is mapped to $X_1^3X_2^2$ and both are in the same representation (the trivial one). ♣

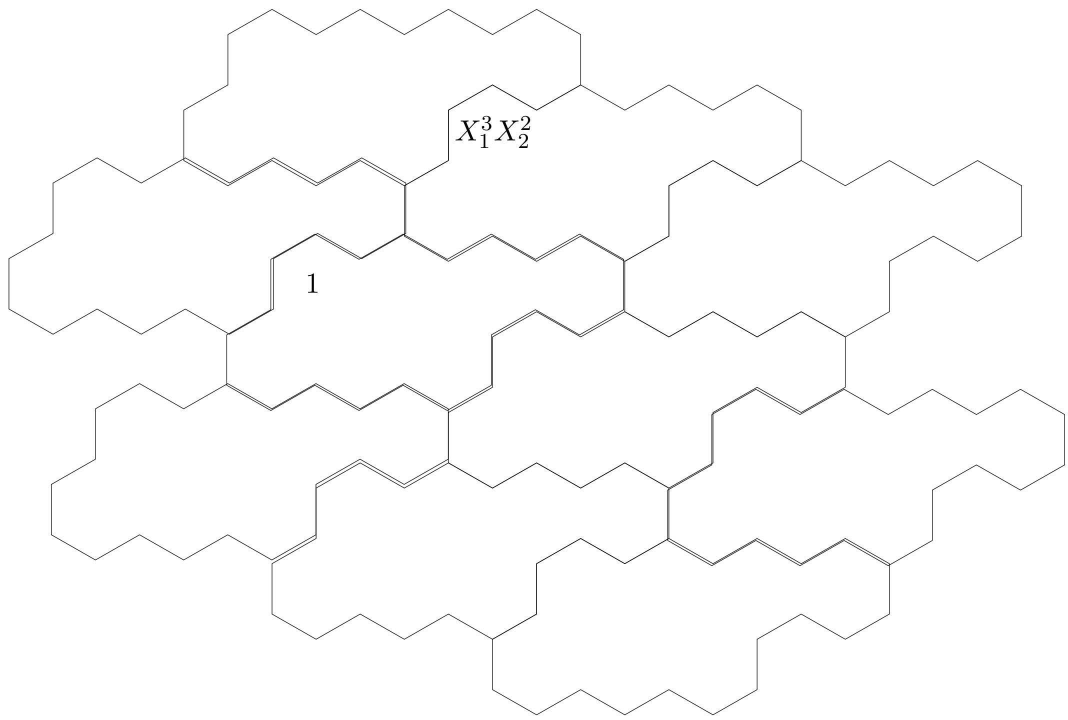

Figure 1.11: Tessellation of the plane by a $G-$graph for H_3.

We end this section by some comments on possible generalizations to higher dimensions. Unfortunately, the Craw-Reid algorithm can not be generalized furthermore. The key-point in this algorithm is that one can reduce to the two-dimensional case, where the Hirzbruch-Jung algorithm can be applied. Consider a subgroup of $SL_4(\mathbb{C})$ of the form $G = 1/r(a_1, a_2, a_3, a_4)$ for some integers r, a_i.

If we want to apply an algorithm similar to the Craw-Reid algorithm, the first step is to take the junior points. This is always possible. The following step is to reduce to the three-dimensional case, by "erasing" in turn i one of the coordinates and considering the action of the resulting groups A_i on the affine space $\mathbb{A}^3$. The problem is that the groups A_i might not be subgroups of $SL_3(\mathbb{C})$, in which case the Craw-Reid algorithm can not be applied. Nevertheless, one can combine the Craw-Reid algorithm and Nakamura's method. Geometrically, as in the Craw-Reid algorithm, the subdivision of the cone σ_0 in order to obtain the $G-\mathrm{Hilb}\mathbb{A}^4$ is given by the subdivision of the junior simplex Δ_4 by help of $\mathrm{Jun}(G)$. See Figure 1.12 for the junior simplex Δ_4 by help of junior points for the group H_4. The description of the cones is given by help of Nakamura's method, this is using H_4-graphs.

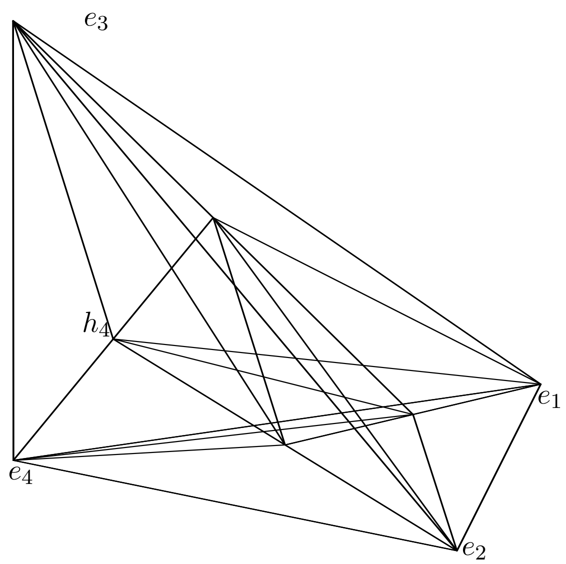

Figure 1.12: Division of Δ_4 for $H_n - \mathrm{Hilb}\mathbb{A}^4, H_n = \frac{1}{15}(1,2,4,8)$

Chapter 2

Generalities on stacks

Introduction: notations and recollections

This first two parts of this chapter are expository. We consider that it is necessary to put together a panoramic view on different approaches for the notion of a stack. This goes as follows. Section 2.1 contains the general definition of a stack, introduced by [13] and developed in [43], [44] and [45]. In section 2.1.2, we recall an equivalent definition of an $S-$stack following [27]. We prove that for an $S-$stack the two definitions coincide (Lemma 2.34). Section 2.1.3 contains some particular classes of stacks such as algebraic stacks, Deligne-Mumford stacks or quotient stacks. The next section gives the link with the algebraic geometry. In 2.2.1, we start by providing the formal definition of a groupoid space. This is not in general an $S-$stack, but the example of a quotient stack can be seen as a groupoid space. In Section 2.2.2, following [26], we relate the formal point of view introduced in 2.2.1 to some problems from algebraic geometry. This also contains the starting point of the sheafification process described in section 2.2.3, following [45].

Section 2.3 makes an attempt to a possible general formalism. In the last part we construct the smooth Deligne-Mumford stack associated to a pair (X, D), with X normal variety and D a fixed $\mathbb{Q}-$divisor. For this construction, we have to provide a precise description of the sheafification (official term stackification) of a certain $2-$functor.

Before starting, we recall some definitions and notations. We denote by $\mathfrak{Cat}$ the category of all categories and by (Set), (Ring), (Group) respectively the categories of all sets, rings, groups. We sometimes call an element of a given set a section. For an object U in a category $\mathcal{C}$ we denote by $h_U = \mathrm{Hom}_{\mathcal{C}}(\cdot, U)$ the contravariant functor

$$\begin{cases} \mathcal{C} & \to & \text{(Set)} \\ Y & \mapsto & \mathrm{Hom}_{\mathcal{C}}(Y, U) \end{cases}$$

If $f : Y \to U$ is an arrow in a category $\mathcal{C}$, we denote by h_f the map associating to any arrow $a : Y' \to Y$ the composition $h_f(a) := f \circ a$. If $\mathcal{C}$ admits fiber products, U is an object of $\mathcal{C}$ and $\{U_i\}_{i\in I}$ is a collection of objects such that for any index i there exist an arrow $a_i : U_i \to U$, we denote by p_i the projection $U_i \times_U U_j \to U_i$ on the factor indexed by i and by p_{ij} the projection $(U_i \times_U U_j) \times_U U_k \to U_i \times_U U_j$.

Let now k be a field. We denote by $\overline{k}$ an algebraic closure of k. If $A \subset B$ is a commutative ring extension we denote by $\overline{A}^B$ the integral closure of A in B. The integral closure of a domain A in its field of fractions $\mathrm{Frac}(A)$ is denoted by A'.

Definition 2.1. *([31], I, §3) Let A be a $k-$algebra. We say that A is separable over k if $A \otimes_k \overline{k}$ has zero Jacobson radical.*

For a scheme X over $\mathrm{Spec}(k)$, we denote by $k(X)$ its field of rational functions.

Definition 2.2. *([31], I, §1) Let X be an integral scheme and L a finite field extension of $k(X)$. The normalization of X in L is a pair (Y, f) where Y is an integral scheme with $k(Y) = L$ and $f : Y \to X$ is an affine morphism such that for any affine open sub-scheme U of X*

$$\Gamma(f^{-1}(U), \mathcal{O}_Y) = \overline{\Gamma(U, \mathcal{O}_X)}^L.$$

Definition 2.3. *([31], I,§3) A morphism of schemes $f : X \to Y$ is unramified at $x \in X$ if $\mathfrak{m}_x = \mathfrak{m}_{f(x)}\mathcal{O}_x$ and $k(x)$ is a finite separable field extension of $k(f(x)) := \mathfrak{m}_{f(x)}/\mathfrak{m}^2_{f(x)}$. We say that f is unramified if it is unramified at any x of X. We denote by $R(f)$ or $\mathrm{Ram}(f)$ the set of points where f is not unramified and call it the ramification locus of f.*

Remark 2.4. (Properties of unramified morphisms, [31], I, Propositions 3.2 and 3.5) If $f : X \to Y$ is locally of finite type, with scheme theoretic fiber $X_y = f^{-1}(y)$, for $y \in Y$, then the following are equivalent:

1. f is unramified;
2. $\forall y \in Y, X_y \to \mathrm{Spec}(k(y))$ is unramified;
3. $\forall y \in Y, X_y$ has an open covering by spectra of finite separable $k(y)-$algebras (and X_y it-self is such a spectra if f is of finite-type; in such a case f has finite fibers, i.e. f is quasi-finite);
4. $\forall y \in Y, X_y$ is $\coprod \mathrm{Spec}(k_i)$, with $k(y) \subset k_i$ finite separable field extension (and this sum is finite if f is of finite-type; if so, f is quasi-finite);
5. $\Omega_{X/Y} = 0$;
6. $\Delta_{X/Y}$ is an open immersion. ♣

Definition 2.5. *([31] I, §3) A morphism of schemes $f : X \to Y$ is etale if it is flat and unramified.*

Remark 2.6. (Jacobian criterion, [26], I, Theorem 4.1) A map $f : X \to Y$ is etale if for any point x of X, there exists an open affine sub-scheme $\mathrm{Spec}(A) \subset X, x \in \mathrm{Spec}(A)$, and an open affine sub-scheme $\mathrm{Spec}(B) \subset Y$, containing $f(x)$, such that $f(\mathrm{Spec}(A)) \subset \mathrm{Spec}(B)$ and the Jacobian condition holds:

$$A = B[X_1, \ldots, X_n]/\langle p_1(X_1, \ldots, X_n), \ldots, p_n(X_1, \ldots, X_n)\rangle, \text{ with}$$

$$\det J(p_1, \ldots, p_n) = \det(\partial p_i/\partial X_j)_{i,j\in\{1,\ldots,n\}} \text{ a unit in } A.$$

From an analytic point of view, this means that we have a local isomorphism at every point of X, but in general this is not true in the algebraic case (see [31], Example 3.4).

An etale morphism is the same as a smooth morphism of relative dimension zero ([18], III, Exercise 10.3). ♣

Remark 2.7. In the literature, a morphism $f : X \to Y$ with $\mathrm{codim}_X(R(f)) > c$, for an integer c is called an morphism etale in codimension c. ♣

Recall 2.8. Finally, we recall the **purity theorem of Zariski-Nagata** ([1], Chapter X,§3, Theorem 3.1):

Let $f : X \to Y$ be a quasi-finite, dominant morphism of integral schemes, with X normal, Y regular and locally noetherian. Let Z be the set of points of X where f is not etale, this is f is unramified (cf. [1], 9.5). Then, if $Z \neq X$, the codimension of Z in X is one at each point, that is for any irreducible component Z' of Z, of generic point z, the Krull dimension of the local ring $O_{X,z}$ is one. ♣

2.1 Stacks in the classical sense

This section is a purely expository one. We give a collection of some known definitions and properties for the notion of a stack. In section 2.1.3 we make a brief recall on algebraic [Deligne-Mumford] stacks, quotient stacks and the relation stack-groupoid space.

2.1.1 Stacks on general sites

We recall here the definition of a stack following [13] and [43]-[44]-[45]. In this section, $\mathcal{C}$ is a category with products and fibered products. The morphisms in this category are also called arrows and denoted by Latin letters a, b, while objects are denoted by capital letters U, V, X, Y.

Definition 2.9. *A category over $\mathcal{C}$ is a category $\mathcal{F}$ equipped with a functor $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$.*

In the sequel, objects of such a category are denoted by u, v and arrows by Greek letters α, β.

Definition-Notation 2.10. *([45], §3.1 "cartesian arrow") Let $\mathcal{F}$ be a category over $\mathcal{C}$. Let u, v be two objects of $\mathcal{F}$ and $\alpha : v \to u$ an arrow between them. We call α cartesian if for any object w of $\mathcal{F}$, any arrow $\gamma : w \to u$ of $\mathcal{F}$ and any arrow $c : p_{\mathcal{F}}w \to p_{\mathcal{F}}v$ of $\mathcal{C}$ with $p_{\mathcal{F}}\alpha \circ c = p_{\mathcal{F}}\gamma$, there exists an unique arrow $\beta : w \to v$ such that $p_{\mathcal{F}}\beta = c$ and $\alpha \circ \beta = \gamma$. In other words, the diagram of Figure 2.1 is commutative. We say that v is a pullback of $p_{\mathcal{F}}u$ along $p_{\mathcal{F}}\alpha$.*

For u an object of $\mathcal{F}$ and $a : V \to U$ an arrow of $\mathcal{C}$ such that $p_{\mathcal{F}}u = U$, a pullback of U along a is also denoted by a^*u.

$$
\begin{tikzcd}
w \arrow[dr, dashed, "\beta"'] \arrow[drr, bend left, "\gamma"] \arrow[dd, mapsto] & & \\
& v \arrow[r, "\alpha"'] \arrow[dd, mapsto] & u \arrow[dd, mapsto] \\
p_{\mathcal{F}}w \arrow[dr, "c=p_{\mathcal{F}}\beta"'] \arrow[drr, bend left] & & \\
& p_{\mathcal{F}}v \arrow[r, "p_{\mathcal{F}}\alpha"'] & p_{\mathcal{F}}u.
\end{tikzcd}
$$

Figure 2.1: Cartesian diagram for cartesian arrow.

Definition 2.11. *([45], §3.5, "fibered category") A fibered category $\mathcal{F}$ over $\mathcal{C}$ is a category over $\mathcal{C}$ such that for any arrow $a : V \to U$ in $\mathcal{C}$ and any object u of $\mathcal{F}$ mapping to U there exists an object v and a cartesian arrow $\alpha : v \to u$ with $p_{\mathcal{F}}\alpha = a$. In other words, we have the existence of "the pullback of any object along any arrow".*

Definition 2.12. *([45], §3.8, "fiber")*

1. *Let $\mathcal{F}$ be a fibered category over $\mathcal{C}$. For U an object of $\mathcal{C}$, the fiber (or the fiber-category) of $\mathcal{F}$ over U denoted by $\mathcal{F}(U)$, is the subcategory of $\mathcal{F}$ with objects the objects u of $\mathcal{F}$ such that $p_{\mathcal{F}}u = U$ and with arrows the arrows α of $\mathcal{F}$ such that $p_{\mathcal{F}}\alpha = \mathrm{id}_U$.*

2. *If each fiber is a groupoid (that is a category in which every arrow is invertible), we say that $\mathcal{F}$ is fibered in groupoids.*

Remark 2.13. Let $\mathcal{F}$ be a category fibered in groupoids. Two pullbacks of u along a map a are isomorphic. In the sequel, we fix arbitrarily such a pullback and denote it by a^*u. Some authors (see [27], 2) call this base-change via a. In general, if $\mathcal{F}$ is not fibered in groupoids, one makes a choice of pull-backs and call the resulting set a cleavage.

By [1], the fiber-category $\mathcal{F}(U)$ is the same as the fiber product $\mathcal{F}\times_{\mathcal{C}}\{U\}$, where $\{U\}$ denotes the subcategory of $\mathcal{C}$ with only one object and the identity arrow. See also [13], section §4 and Notation 2.30 below. ♣

Lemma 2.14. *Let $\mathcal{F}$ be a fibered category over $\mathcal{C}$, U an object of $\mathcal{C}$ and u an object in $\mathcal{F}$ with $p_{\mathcal{F}}u = U$. Then any arrow $a : V \to U$ of $\mathcal{C}$ gives an order-reversing functor $a^* : \mathcal{F}(U) \to \mathcal{F}(V)$ between the fibers over U and V. Moreover, for any arrow $b : W \to V$ there is a canonical isomorphism between the functors $(ab)^*$ and b^*a^*.*

Proof:

Take a pullback a^*u of U along a as in Remark 2.13. We define a functor $a^* : \mathcal{F}(U) \to \mathcal{F}(V)$ by sending each object u of $\mathcal{F}(U)$ to a^*u and each arrow $\beta : u \to v$ of $\mathcal{F}(U)$ to the unique arrow $a^*\beta : a^*u \to a^*v$ of $\mathcal{F}(V)$ such that the diagram of Figure 2.2

$$\begin{tikzcd} a^*u \arrow[r] \arrow[d, dashed, "a^*\beta"] & u \arrow[d, "\beta"] \\ a^*v \arrow[r] & v \end{tikzcd}$$

Figure 2.2: Commutative square for pull-backs.

is commutative. The isomorphism between the functors $(ab)^*$ and b^*a^* comes from the fact that both give pullbacks of the same object of the fiber $\mathcal{F}(W)$. ∎

Definition 2.15. *([44], Definition 1.16, "Grothendieck pretopology") Let $\mathcal{C}$ be a category with fibered products. A Grothendieck pretopology on $\mathcal{C}$ is given as follows. For any object U of $\mathcal{C}$ give a collection of sets of arrows $\{U_i \to U\}$, called a covering of U, where the following conditions are satisfied:*

1. *if $U' \to U$ is an isomorphism, then the set with one map $\{U' \to U\}$ is a covering.*

2. *if $\{U_i \to U\}$ is a covering and $V \to U$ is an arrow, then the collection of projections from the fibered products on the second term $\{U_i \times_U V \to V\}$ is also a covering.*

3. *if $\{U_i \to U\}$ is a covering and $\{W_{ij} \to U_i\}$ is a covering of U_i for any index i, then the collection of compositions $\{W_{ij} \to U\}$ is also a covering.*

Remark 2.16. In [2], Verdier gives the definition of a Grothendieck topology (Definition 1.1, page 219) in terms of sieves (fr. "cribles", Definition 4.1, page 20. This goes as follows. For an object U of $\mathcal{C}$ he defines a sieve to be a sub-functor of $h_U = \mathrm{Hom}(\cdot, U)$. This is the same as to give a collection of arrows on U, $\mathcal{S}(U) := \{T \to U\}$ such that every composition $T' \to T \to U$ is still in $\mathcal{S}(U)$. We notice that any covering $\mathcal{U} = \{U_i \to U\}$ in the sense of 2.15, gives a sieve via the functor

$$\begin{cases} h_{\mathcal{U}} : \mathcal{C} & \to & (\mathrm{Set}) \\ \quad V & \mapsto & \{V \to U / \exists i \text{ with } V \to U_i \to U\} \end{cases}$$

Now, for any object U of $\mathcal{C}$ fix a set $\mathcal{S}(U)$ and impose some conditions on this collection of arrows (stable by base-change, local character and identity-map as in Definition 1.1, page 219, op. cit.). Then, the collection of all $\mathcal{S}(U), U$ object of $\mathcal{C}$, is called a topology on $\mathcal{C}$. Say this is etale if $\mathcal{C}$ is the category of schemes [over a base scheme] and all the arrows are etale. By Remark 1.3.1 op.cit., a Grothendieck pretopology defines a Grothendieck topology.

Two different Grothendieck pretopologies may define the same Grothendieck topology. According to [44], Proposition 2.46, for two different Grothendieck pretopologies $\mathcal{T}$ and $\mathcal{T}'$ to define the same Grothendieck topology it is enough that every covering in $\mathcal{T}$ is also in $\mathcal{T}'$ and that every covering in $\mathcal{T}'$ has a refinement in $\mathcal{T}$. Here, a refinement of a covering is like in Definition 2.17.

So, in a certain sense, it is enough to work with a Grothendieck pretopology instead of a Grothendieck topology, which is more natural from an analytic point of view, while Verdier's definition is more natural from the point of view of categorical algebra. ♣

Definition 2.17. *([45], Definition 2.44 "refinement") Let $\mathcal{C}$ be a category and $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ a set of arrows. We say that the set of arrows $\mathcal{V} = \{g_a : V_a \to U\}_{a \in A}$ is a refinement of $\mathcal{U}$ if:*

$$\forall a \in A \quad \exists i(a) \in I \text{ and } \exists h_a : V_a \to U_{i(a)} \text{ such that } f_{i(a)} h_a = g_a.$$

Definition 2.18. *A category $\mathcal{C}$ with a Grothendieck topology $\mathcal{T}$ is called a site and it is denoted by $(\mathcal{C}, \mathcal{T})$.*

Example 2.19. The classical example of a site is the etale topology (see also [27], 1). Fix S a base-scheme. Denote by (Sch/S) (respectively by (Aff/S)) the category of (affine) schemes over S. For any scheme U over S, a covering family is a finite collection of etale morphisms $\{f_i : V_i \to U\}$ whose images $U_i := f_i(V_i)$ are open sets that cover U. ♣

Remark 2.20. We notice in the example above that taking the union of the sets U_i gives a covering family of U with only one element. By Remark 2.16, this is enough for defining the etale topology on (Sch/S). ♣

Remark 2.21. ([26], Chapter I, Proposition 4.9 and 4.10) The following properties are stable in the etale topology:

1. for schemes: locally noetherian, reduced, normal, nonsingular, of dimension n over a ground field;

2. for morphisms: quasicompact, [quasi]separated, universally closed, of finite type, of finite presentation, [quasi]finite, isomorphism.

Thus, for a map $f : X \to Y$ between schemes, the singular locus of X, $\mathrm{Sing}(X)$ is contained in $R(f) = \mathrm{Ram}(f)$, the ramification locus of f. ♣

Example 2.22.
1. Other classical examples of topologies on (Sch/S) are the fppf topology and the fpqc topology (see [27], Section 9). The fppf (fr. "fidèlement plate de présentation finie") topology is defined by the pretopology where a covering for an $S-$scheme U is a family/collection of flat morphisms, locally of finite presentation whose images cover U. The fpqc (fr. "fidèlement plate quasi-compact") topology is defined by flat morphisms. In the literature, one denotes by TOP one of the three topologies – etale, fppf or fpqc – on (Sch/S) (or (Aff/S)).

2. Another example of a site is the site associated to a topological space X. The topological space X gives a category with objects open subsets and arrows inclusions. The associated Grothendieck topology is the one for which a covering for an open set is just an open covering in the classical sense.

♣

Definition 2.23. *([44], Definition 3.2, "descent data/object with descent data/descent data on an object ") Let $\mathcal{F}$ be a fibered category over a given site $\mathcal{C}$. Let U be an object of $\mathcal{C}$ and $\mathcal{U} := \{a_i : U_i \to U\}$ be a covering of U in $\mathcal{C}$. We denote by U_{ij} the fiber product $U_i \times_U U_j$.*

A descent data on $\mathcal{U}$ is a pair $(\{u_i\}, \{\alpha_{ij}\})$, where u_i is an object of the fibers $\mathcal{F}(U_i)$ and α_{ij} is an isomorphism in $\mathcal{F}(U_{ij})$ between $\mathrm{p}_j^* u_j$ and $\mathrm{p}_i^* u_i$, such that the following **cocycle condition** *is satisfied:*

$$p_{ik}^* \alpha_{ik} = p_{ij}^* \alpha_{ij} \circ p_{jk}^* \alpha_{jk} : p_k^* u_k \to \mathrm{p}_i^* u_i, \forall i, j, k. \tag{2.1.1}$$

Here p_{st} and p_s are the projections on the factor indexed by st, respectively by s, from $U_i \times_U U_j \times_U U_k$.

Lemma 2.24. *Let $\mathcal{F}$ be a fibered category over a site $\mathcal{C}$, U an object of $\mathcal{C}$ and fix $\mathcal{U} := \{a_i : U_i \to U\}$ a covering of U.*

1. *The descent data on $\mathcal{U}$ form a category. We denote it by $\mathcal{F}(\mathcal{U})$ and call it a descent category (on U).*

2. *There is a functor between the fiber $\mathcal{F}(U)$ and the descent category $\mathcal{F}(\mathcal{U})$.*

Proof:

1. Define a morphism between two descent data $(\{u_i\}, \{\alpha_{ij}\})$ and $(\{v_i\}, \{\beta_{ij}\})$ "coordinate by coordinate". We fix an index i. A morphism between u_i and v_i is an arrow γ_i in the fiber-category $\mathcal{F}(U_i)$, such that the following condition is satisfied. Apply the pullback-functors of Lemma 2.14 for the projections $p_i : U_i \times_U U_j \to U_i$ and $p_j : U_i \times_U U_j \to U_j$ for the arrows γ_i and γ_j. We ask that the diagram of Figure 2.3 bellow commutes. With this definitions, all the axioms of a category are satisfied so that $\mathcal{F}(\mathcal{U})$ forms a category.

$$\begin{tikzcd} p_j^* u_j \arrow[r, "p_j^*\gamma_j"] \arrow[d, "\alpha_{ij}"'] & p_j^* v_j \arrow[d, "\beta_{ij}"] \\ p_i^* u_i \arrow[r, "p_i^*\gamma_i"] & p_i^* v_i \end{tikzcd}$$

Figure 2.3: Morphism between descent data

2. Let u be an object of $\mathcal{F}(U)$. We associate to u the following descent data on $\mathcal{U}$. The objects u_i are the pullbacks $a_i^* u$ and the isomorphisms α_{ij} between $p_j^* a_j^* u$ and $p_i^* a_i^* u$ come from the fact that both $p_j^* a_j^* u$ and $p_i^* a_i^* u$ are the pullbacks of u to $U_i \times_U U_j$.

   For an arrow $\alpha : v \to u$ of $\mathcal{F}(U)$, we get for each index i a pullback arrow $a_i^* \alpha : a_i^* v \to a_i^* u$, as in Lemma 2.14. This gives an arrow from the descent data associated to v to the one associated to u. ■

We can now introduce the definition of a stack.

Definition 2.25. *([45], Definition 4.6, "stack")*
Let $\mathcal{F}$ be a fibered category on a site $\mathcal{C}$.

1. *We say that $\mathcal{F}$ is a prestack over $\mathcal{C}$ if for each covering $\mathcal{U} := \{U_i \to U\}$ of $\mathcal{C}$, the functor $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ is fully faithful.*

2. *We say that $\mathcal{F}$ is a stack over $\mathcal{C}$ if for each covering $\mathcal{U} := \{U_i \to U\}$ of $\mathcal{C}$, the functor $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ is an equivalence of categories.*

Notation 2.26. Let S be a scheme. A stack on the etale site (Aff/S) (or (Sch/S)) of Example 2.19 is called an $S-$stack. We denote by (St/S) the category of $S-$stacks. ♣

Before passing to the next section, let us introduce the definition of a sheaf on a category.

Definition 2.27. *Let $\mathcal{C}$ be a site.*

1. *A contravariant functor $F : \mathcal{C} \to (\mathrm{Set})$ is called a presheaf.*
2. *A presheaf $F : \mathcal{C} \to (\mathrm{Set})$ is called a sheaf on sets over $\mathcal{C}$ if the following two conditions are satisfied:*
   (a) *("separated") for any object U of $\mathcal{C}$, any covering $\{a_i : U_i \to U\}$ and any sections u and v in $F(U)$ such that $Fa_i(u) = Fa_i(v)$ for all i, it follows that $u = v$.*
   (b) *("effective") for any object U of $\mathcal{C}$, any covering $\{a_i : U_i \to U\}$ and any sections u_i of $F(U_i)$ with*
   $$Fp_i u_i = Fp_j u_j \text{ in } F(U_i \times_U U_j), \forall i, j,$$
   there exists a unique section u of $F(U)$ such that $Fa_i(u) = u_i$, for all i.
3. *A morphism of sheaves is a natural transformation of functors.*

Remark 2.28. We say that a sheaf is a sheaf on groups (Group), rings (Ring), modules (Mod), or more generally on a small category if the composition with the forgetful functor to the category (Set) is a sheaf on sets.

The most general target-category for a presheaf F is the category of all categories $\mathfrak{Cat}$, seen as a $1-$category. There is no natural way to define the notion of a sheaf on $\mathfrak{Cat}$ by help of forgetful functors. Thus, the necessity of the constructions and definitions of Section 2.3. ♣

Notation 2.29. Let S be a scheme. A sheaf on (Aff/S) (or (Sch/S)) is called an $S-$space. Obviously, a sheaf on (Aff/S) can be extended to a sheaf on (Sch/S). The $2-$category of all $S-$spaces is denoted by (Sp/S). ♣

In the language of $2-$categories, a functor between two categories is a $1-$arrow and a natural transformation between two functors is called a $2-$arrow. So a sheaf is a $1-$arrow and a morphism of sheaves is a $2-$arrow.

2.1.2 $S-$stacks – another approach

In this sequel we fix S a scheme and treat the example of an $S-$stack from the point of view of sheaves, as defined in [27]. This approach is interesting especially because it gives the relations between various categories. All the previous notations and conventions hold.

Notation 2.30. (cf.[27], Definition 2.1, "$S-$groupoid") An $S-$groupoid is a category fibered in groupoids over the etale site (Aff/S) (or (Sch/S)). We denote by (Gr/S) the $2-$category of all $S-$groupoids. ♣

Remark 2.31. The category (Sch/S) is a full sub-category of (Sp/S) : a scheme U over S gives the $S-$space $\mathrm{Hom}_{(\mathrm{Sch}/S)}(\cdot, U)$. The $2-$category (Sp/S) is a full sub$-2-$category of the $2-$category of $S-$groupoids (Gr/S). To an $S-$space F we associate the $S-$groupoid with objects pairs (U, u) where U is a scheme over S and u a section of $F(U)$ and arrows $(V, v) \to (U, u)$ given by a morphism $f : V \to U$ such that $F(f)(u) = v$. The associated functor $p_F : F \to (\mathrm{Aff}/S)$ sends (U, u) to U. ♣

Definition 2.32. *([27], Definition 3.1, "$S-$stack") Let $\mathcal{F}$ be an $S-$groupoid. We say that it is an $S-$stack if the following are satisfied:*

1. *("sheaf") for any $S-$scheme U and any objects u and v in the fiber $\mathcal{F}(U)$, the functor*

$$\begin{cases} (\mathrm{Aff}/U) & \to & (\mathrm{Set}) \\ (a : V \to U) & \mapsto & \mathrm{Hom}_{\mathcal{F}(V)}(a^*u, a^*v) \end{cases}$$

   is an $S-$space.

2. *("effective") any descent data on an $S-$scheme U is effective: given any descent data $(\{u_i\}, \{\alpha_{ij}\})$ on any covering $\mathcal{U} := \{U_i \to U\}$ there exists an object u in the fiber $\mathcal{F}(U)$ with isomorphisms $f_i : a_i^*u \simeq u_i$ for each index i.*

We denote by (St/S) the $2-$category of $S-$stacks.

Remark 2.33. The sheaf-condition 1 of the previous definition implies that the object u endowed with the family (f_i) of 2 is unique up to canonical isomorphism. (St/S) is a full sub$-2-$category of (Gr/S). ♣

Lemma 2.34. *Definition 2.25 and Definition 2.32 for an $S-$stack, are equivalent.*

Proof:

Let $\mathcal{F}$ be a fibered category on the etale site of (Aff/S) (or (Sch/S)). In the proof, we denote by U a scheme over S and by$\mathcal{U} := \{a_i : U_i \to U\}$ an etale covering of U.

We suppose that 1 and 2 of Definition 2.32 hold. We want to prove that for any $S-$scheme U the functor of Lemma 2.24, (2) between the fiber-category and the descent-category on U is an equivalence. We use the effectiveness condition of Definition 2.32, (2b) to define an inverse: to a descent data $(\{u_i\}, \{\alpha_{ij}\})$ associate the object u of the fiber $\mathcal{F}(U)$ such that there exist an isomorphisms $a_i^* u \simeq u_i$, well defined up to unique isomorphism by Remark 2.33.

We suppose now that $\mathcal{F}$ is an $S-$stack as in (2) of Definition 2.25 and we denote by T the equivalence $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$. We want to prove (1) and (2) of Definition 2.32. To prove 2.32, (2), to a descent data $(\{u_i\}, \{\alpha_{ij}\})$ of $\mathcal{F}(\mathcal{U})$ we associate $u = T(\{u_i\}, \{\alpha_{ij}\})$. Then, the isomorphisms $a_i^* u \simeq u$ follows by applying T^{-1} to u, where T^{-1} is an inverse of T.

We prove now that the sheaf-condition holds. Let U be a scheme over S and u and v two objects in the corresponding fiber $\mathcal{F}(U)$. We need to prove that the functor F bellows is an $S-$space.

$$\begin{array}{rcc} F : (\mathrm{Aff}/U) & \to & (\mathrm{Set}) \\ (a : V \to U) & \mapsto & \mathrm{Hom}_{\mathcal{F}(V)}(a^* u, a^* v) \end{array}$$

We first prove that F is a $U-$space, that is it is separated and effective in the sense of Definition 2.27. Let $a : V \to U$ be a $U-$scheme. We fix $\mathcal{V} := \{b_i : V_i \to V\}$ a covering of V. We denote by a_i the arrow $a \circ b_i : V_i \to U$, for all i. Then, $F(V) = \mathrm{Hom}_{\mathcal{F}(V)}(a^* u, a^* v)$ and $F(V_i) = \mathrm{Hom}_{\mathcal{F}(V_i)}(a_i^* u, a_i^* v)$. By Lemma 2.14, a_i^* is the same as $b_i^* a^*$ and we can associate to $\{a_i^* u\}$ (respectively to $\{a_i^* v\}$) in a canonical way a descent data: the isomorphisms α_{ij} (respectively β_{ij}) come from the fact that $a_i^* u$ and $a_j^* u$ are both pullbacks of the same object u (respectively v). If x is a section of $F(V)$, that is an arrow between $a^* u$ and $a^* v$ in the fiber $\mathcal{F}(V)$, then $F b_i(x)$ equals $b_i^* x$.

To prove F is separated, let x and y be two sections of $F(V)$ such that $b_i^* x = b_i^* y$, for all i. Then, in the descent-category $\mathcal{F}(\mathcal{V})$, the arrows x and y define the same arrow between descent data associated to $\{a_i^* u\}$ and $\{a_i^* v\}$. Because $\mathcal{F}(V) \simeq \mathcal{F}(\mathcal{V})$, it follows that the corresponding arrows of $\mathcal{F}(V)$ are equal, that is $x = y$.

To prove that F is effective in the sense of Definition 2.27, (2b), we consider sections x_i in $FV_i = \mathrm{Hom}_{\mathcal{F}(V_i)}(a_i^* u, a_i^* v)$ such that $F p_i x_i = F p_j x_j$, for all indices i and j. This is equivalent to say that $p_i^* x_i = p_j^* x_j$, for all i and j. In particular, the cocycle condition is satisfied, so that we can associate to $\{x_i\}$ an arrow in $\mathcal{F}(\mathcal{V})$. It follows from the equivalence $\mathcal{F}(V) \simeq \mathcal{F}(\mathcal{V})$, that there exists an unique arrow x in $\mathcal{F}(V)$ such that each x_i is the pullback of x along a_i.

Now, to prove that F is an $S-$space: we take the base-change functor

$$\begin{array}{rcc} \tilde{F} : (\mathrm{Aff}/S) & \to & (\mathrm{Set}) \\ V & \mapsto & F(V \times_S U) \end{array}$$

which is an $S-$space, by the above argument. ∎

Remark 2.35. Let F be an $S-$space. We give at the beginning of the section, in Remark 2.31, that any $S-$space is an $S-$groupoid. In fact, it is true that any $S-$space is an $S-$stack (cf. [27], 3.4.1). Thus, (Sp/S) is a full sub$-2-$category of (St/S). ♣

2.1.3 Examples

As in the section above, let S be a scheme. Following [27], we recall the definitions of an algebraic stack, a Deligne-Mumford stack and the example of a quotient stack. We start with the definition of fiber product and diagonal morphism.

Construction 2.36. ([27], 2.2.2, "fiber product for groupoids") Let $\mathcal{F}, \mathcal{G}$ and $\mathcal{V}$ be three $S-$groupoids and $F : \mathcal{F} \to \mathcal{V}, G : \mathcal{G} \to \mathcal{V}$ two $1-$arrows. The fiber product of $\mathcal{F}$ and $\mathcal{G}$ over $\mathcal{V}$, via F and G is an $S-$groupoid denoted $\mathcal{F} \times_{F,\mathcal{V},G} \mathcal{G}$ defined as follows. For any $S-$scheme U the fiber-category has objects the triples (u, v, a) where u is an object in the fiber $\mathcal{F}(U)$, v is an object in the fiber $\mathcal{G}(U)$ and a is an arrow from $F(u)$ to $G(v)$ in the fiber $\mathcal{V}(U)$. An arrow between two triples (u_1, v_1, a_1) and (u_2, v_2, a_2) is a pair $(\alpha : u_1 \to u_2, \beta : v_1 \to v_2)$ such that $a_2 \circ F(\alpha) = G(\beta) \circ a_1$. For any arrow $f : V \to U$ of $S-$schemes we define in the obvious way the base-change functor f^* between the corresponding fibers of the fiber product. The fiber product is an $S-$groupoid. The same construction can be done for $S-$stacks or $S-$spaces with the result being an $S-$stack or an $S-$space. ♣

Construction 2.37. ([27], Definition 2.2.3, "diagonal morphism") Let $\mathcal{F}, \mathcal{G}$ be two $S-$groupoids and $F : \mathcal{F} \to \mathcal{G}$ a $1-$morphism between them. We define the diagonal morphism $\Delta_{\mathcal{F}/\mathcal{G}} : \mathcal{F} \to \mathcal{F} \times_{F,\mathcal{G},F} \mathcal{F}$ on each fiber. For U an $S-$scheme and u an object in $\mathcal{F}(U)$, we put $\Delta_{\mathcal{F}/\mathcal{G}}(u) = (u, u, \mathrm{id}_{F(u)})$. For an arrow $\alpha : u \to v$ of the fiber, we put $\Delta_{\mathcal{F}/\mathcal{G}}(\alpha) = (\alpha, \alpha)$. ♣

2.1.3.1 Algebraic stacks and Deligne-Mumford stacks

We start by recalling some of the properties of $1-$arrows between $S-$spaces and $S-$stacks.

Definition 2.38. 1. *([27], §1, "scheme-like $1-$arrow between $S-$spaces") A morphism $p : F \to G$ in (Sp/S) is said to be scheme-like if for any $S-$scheme U and any section u of $G(U)$ (seen as an arrow between $S-$spaces $u : U \to G$ via*

$$\begin{cases} \mathrm{Hom}_S(V, U) & \to & G(V) \\ (f : V \to U) & \mapsto & G(f)(u) \end{cases}$$

for any $S-$scheme V) the fiber product $U \times_{u,G,p} F$ is an $S-$scheme. We call $U \times_{u,G,p} F$ a base-change.

$$\begin{tikzcd} \underset{\text{scheme}}{U\times_{u,G,p}F} \arrow[r] \arrow[d] & \underset{\text{scheme}}{U} \arrow[d, "u"] \\ F \arrow[r, "p"] & G \end{tikzcd}$$

Figure 2.4: Defining property for scheme-like $1-$arrow.

2. *We say that a scheme-like morphism has a property $\mathcal{P}$ if for all $S-$schemes U the corresponding base-change of $S-$schemes $U \times_{u,G,p} F \to U$ has the property $\mathcal{P}$.*

Remark 2.39. For an $S-$space F the fact that the diagonal morphism $\Delta : F \to F \times_S F$ is scheme-like implies automatically that any $1-$arrow $X \to F$, with X scheme, is scheme-like. The same holds for stacks (see for example [43], 7.13). ♣

Definition 2.40. *([27], Definition 1.1, "algebraic $S-$space") An $S-$space F is said to be algebraic if the following hold:*

1. *the diagonal morphism $F \to F \times_S F$ is scheme-like and quasi-compact.*

2. *there exists an $S-$scheme X and $1-$arrow $P : X \to F$ etale and onto.*

We denote by (ASp/S) *the full sub-category of* (Sp/S) *whose objects are algebraic $S-$spaces.*

Definition 2.41. *([27], Definitions 1.5 and 3.9)*

1. *("representable $1-$arrow between $S-$spaces") A $1-$arrow of $S-$spaces $p : F \to G$ is representable if for any $S-$scheme U and any section v of $G(U)$ the $S-$space $F \times_{f,G,v} U$ is an algebraic $S-$space.*

2. *("representable $S-$stack") An $S-$stack $\mathcal{F}$ is representable if there exists an algebraic $S-$space F and a $1-$isomorphism of $S-$stacks $F \simeq \mathcal{F}$.*

3. *("representable/scheme-like $1-$arrow between $S-$stacks") A $1-$arrow of $S-$stacks $f : \mathcal{F} \to \mathcal{G}$ is said to be representable (respectively scheme-like) if for any $S-$scheme U and any object u of the fiber $\mathcal{G}(U)$ seen as $1-$arrow from U to $\mathcal{G}$, the fiber product $U \times_{u,\mathcal{G},F} \mathcal{F}$ is representable (respectively representable by an $S-$scheme).*

Remark 2.42. In the above definition, the $1-$arrow from U to $\mathcal{G}$ is defined via the pull-back, on object, as well as on arrows, as in Lemma 2.14. As in Remark 2.39, for a given stack $\mathcal{F}$, if the diagonal morphism is representable, then any $1-$arrow $F \to \mathcal{F}$, with F an algebraic $S-$space is representable (for a proof, see [27], Corollary 3.13). ♣

Definition 2.43. *([27], Definition 4.1 and Remark 4.7.1, "algebraic stack")*

1. *An algebraic $S-$stack is an $S-$stack $\mathcal{F}$ such that:*

   (a) *the diagonal morphism of $S-$stacks*

   $$\Delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$$

   is representable, separated and quasi-compact.

   (b) *there exists an algebraic $S-$space F and a $1-$arrow of $S-$stacks $P : F \to \mathcal{F}$ onto and smooth. We call such a P a presentation of $\mathcal{F}$.*

   We denote by (ASt/S) the full sub$-2-$category of (St/S) whose objects are algebraic $S-$stacks.

2. *We say that an algebraic $S-$stack $\mathcal{F}$ has a property $\mathcal{P}$ if for a/any presentation $P : X \to \mathcal{F}$ the algebraic $S-$space F has the property $\mathcal{P}$.*

As an example of properties as in the definition above: regular, noetherian, Cohen-Macaulay, quasi-compact, smooth...

Definition 2.44. *([27], Definition 4.1, "Deligne-Mumford stack") A Deligne-Mumford stack is an algebraic $S-$stack with an etale presentation. We denote by (DM/S) the category of Deligne-Mumford stacks.*

Example 2.45. ([27] Example 4.1.1 or [43] Example 7.16) Any $S-$stack associated to an algebraic $S-$space is a Deligne-Mumford stack. The diagonal morphism of such a stack is of finite type, unramified, quasi-finite, quasi-affine and representable. The relations between different categories introduced until now are given in Figure 2.5. ♣

$$\begin{tikzcd}
 & (\mathrm{ASp}/S) \arrow[rr, hook] \arrow[dd, hook] \arrow[dr, hook] & & (\mathrm{Sp}/S) \arrow[dd, hook] \\
(\mathrm{Sch}/S) \arrow[ur, hook] \arrow[dr, hook] & & (\mathrm{DM}/S) \arrow[dl, hook] & \\
 & (\mathrm{ASt}/S) \arrow[rr, hook] & & (\mathrm{St}/S)
\end{tikzcd}$$

Figure 2.5: Relation between the category of schemes, [algebraic] spaces and [algebraic] stacks.

2.1.3.2 Quotient by group action

We give here the example of the quotient-stack for $S-$spaces, cf. [27]. For the particular case of an $S-$scheme, the notion was introduced in [13].

Definition 2.46. *Let X be an $S-$space.*

1. *We say that Y is an $X-$space if it is an $S-$space with a $1-$arrow $Y \to X$ such that we have a commutative diagram as in Figure 2.6.*

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

Figure 2.6: $X-$space condition for Y.

2. *We say that an $X-$space G is an $X-$space in groups if G is an $S-$space in groups. If Y is another $X-$space, we say that G acts on Y (on the right) if for any $S-$scheme U there is an action (on the right) of the group $G(U)$ on the set $Y(U)$, such that there is a compatibility between actions in a natural sense.*

Definition 2.47. *([27], 2.4.2) Let U be an $S-$scheme and G a $U-$space on groups. A $G-$torsor (on the right) is a $U-$space T with an action of G (on the right) such that there is a covering family [with one element] $V \to U$ in (Aff/S) such that $T \times_U V$ is $G \times_U V-$isomorphic with $G \times_U V$ endowed with the action of $G \times_U V$ by translation on the right.*

Construction 2.48. With the notations above, let X be an $S-$space, Y a $X-$space and G a $X-$space on groups acting on the right on Y. We construct an $S-$groupoid denoted $[Y/G/X]$ as follows. For any $S-$scheme U we describe the fiber-category $[Y/G/X](U)$. The objects of the fiber are triples (x, T, a) where x is an element of $X(U)$, T is a $(G \times_{X,x} U)-$torsor (as in Figure 2.7) and $a : T \to Y \times_{X,x} U$ is a $(G \times_{X,x} U)-$equivariant morphism of $U-$spaces.

Here, the section x is seen as an arrow from U to X as is Definition 2.38. An arrow in the fiber $[Y/G/X](U)$ between the triple (x, T, a) and (x', T', a') is defined in the following way. See x and x' as functors between the $S-$space associated to U and the $S-$space X. Let M be a functor between x and x' (this is actually a $2-$arrow) and define an arrow between T and T' (respectively a and a') by help of M.

When Y is the variety X it-self we denote by $B(G/X)$ the $S-$groupoid $[X/G/X]$ and call it the classifying of G/X. In this case, for any $S-$scheme U the fiber category is the category of $(G \times_{X,x} U)-$torsors over U. ♣

$$\begin{tikzcd}
(G \times_{X,x} U) \times_U V \arrow[r, "\sim"] \arrow[dd] \arrow[drr] & T \times_U V \arrow[dd] \arrow[dr] & \\
 & & V \arrow[dd] \\
G \times_{X,x} U \arrow[dd] \arrow[drr] & T \arrow[dr] & \\
 & & U \arrow[d, "x"] \\
G \arrow[rr] & & X
\end{tikzcd}$$

Figure 2.7: $(G \times_{X,x} U)-$torsor

Remark 2.49. ([27] Remarks 3.4.2 and 4.6.1) The groupoid $[Y/G/X]$ is an $S-$stack, so in particular $B(G/X)$ is also an $S-$stack. If X is an algebraic $S-$space and G a group-scheme over S, smooth, separated and of finite presentation, acting on the right on X, the quotient stack $[X/G/S]$ is an algebraic $S-$stack. If G is etale, then $[X/G/S]$ is a Deligne-Mumford stack. Moreover the projection $\pi : X \to [X/G/S]$ is a presentation (an etale one for the case of etale G). The product $X \times_{\pi,\mathcal{V},\pi} X$ is representable by $X \times_S G$. The second projection is nothing else but the action of G. ♣

2.2 Stacks via groupoid spaces

In the sequel, we recall some classical results on groupoid spaces. The notations of this section are slightly different from the ones before; we use them in accordance to the standard notations in the literature. From the point of view of categorical algebra, the idea is the following. Let $\mathcal{C}$ be a small category and denote by $U := \text{Ob}\mathcal{C}$ the set of objects and by $R := \text{Mor}\mathcal{C}$ the set of arrows. The axioms of categories give four maps of sets: p and q are the source and target for an arrow, e the identity morphism and m is the composition of arrows.

$$\begin{tikzcd} R \arrow[r, shift left, "q"] \arrow[r, shift right, "p"'] & U \arrow[r, "e"] & R \end{tikzcd} \qquad \qquad R \times_{p,U,q} R \xrightarrow{m} R$$

If $\mathcal{C}$ is a groupoid, then each morphism has an inverse, so there exists also an arrow $i : R \to R$. One can easily heck that in this case the maps p and q induce an equivalence relation on the set U.

Conversely, any equivalence relation R on a set U provide a groupoid. The set of objects is U, the set of arrows is R, the reflexivity gives the identity, the transitivity gives the composition and the symmetry gives the inverse of a map.

The idea is to generalize this to the case of $S-$spaces, for S a fixed scheme. In what follows, we denote by $\mathcal{C}$ a small category with fiber products. A Grothendieck topology on $\mathcal{C}$ is denoted by $\mathcal{T}$.

2.2.1 Definition

The initial construction is given in [13] and outlined in [27]. For recent results and modern formalism on the topic, see [43], 7 and [4], Chapter 4.

Definition 2.50. *([27], Definition 2.4.3, "groupoid $S-$space") Let U and R be two $S-$spaces with five arrows : source $p : R \to U$, target $q : R \to U$, identity $e : U \to R$, inverse $i : R \to R$ and composition $m : R \times_{p,U,q} R \to R$ such that:*

1. $p \circ e = q \circ e = \mathrm{id}_U, p \circ i = q, q \circ i = p, p \circ m = p \circ p_2, q \circ m = q \circ p_1$, *where p_1 and p_2 are the first, respectively the second projection from the fiber product $R \times_{p,U,q} R$ to R,*
2. *(associativity)* $m \circ (m \times \mathrm{id}_R) = (\mathrm{id}_R \times m) \circ m$,
3. *(identity)* $m \circ (e \times \mathrm{id}_R) = m \circ (\mathrm{id}_R \times e) = \mathrm{id}_R$,
4. *(inverse)* $e \circ p = m \circ (i \times \mathrm{id}_R)$ *and* $m \circ (\mathrm{id}_R \times i) = e \circ q$.

We call the pair (U, R) a groupoid space over S or a groupoid $S-$space and denote it by $[U/R]$.

Construction 2.51. ($S-$groupoid associated to a groupoid $S-$space) To such a groupoid $S-$space one can associate an $S-$groupoid denoted $[U, R]'$ (see Notation 2.30 for the notion of $S-$groupoid). For an $S-$scheme X, define the fiber $[U, R]'(X)$ by taking the set of objects to be $U(X)$ and the set of arrows to be $R(X)$. The source, target and composition in the fiber are induced by p, q and respectively m. Any arrow $a : Y \to X$ induces an arrow $a^* : [U, R]'(X) \to [U/R]'(Y)$ in a natural way. There is a projection arrow $\pi : U \to [U, R]'$ given by $U(X) \to \mathrm{Ob}[U, R]'(X)$. The $S-$groupoid $[U, R]'$ is in general only a prestack. We can take the stack associated to it and denote it by $[U, R]$. ♣

Definition 2.52. *Let (U, R, p, q, m, e, i) and $(U', R', p', q', m', e', i')$ be two groupoid spaces. A morphism between them is pair $(a : U \to U', b : R \to R')$ such that:*

$$p' \circ b = a \circ p, q' \circ b = a \circ q, e' \circ a = b \circ e, m' \circ (b \times b) = b \circ m, i' \circ b = b \circ i.$$

Example 2.53. To any arrow $X \to S$ we associate a groupoid space where $U = X, R = X \times_S X, p$ and q the projections, e the diagonal map and i the switching-arrow. If we identify $R \times_{p,S,q} R$ with $X \times_S X \times_S X$, then m is the projection p_{13} onto the first and the third factor. ♣

2.2.2 Algebraic spaces as quotient sheaves by equivalence relation

The formalism of 2.2.1 has its roots in a more down-to-earth idea related to schemes and due to Grothendieck [17]. In what follows, we consider the point of view of [26] for the case of algebraic spaces. The previous conventions for the category $\mathcal{C}$ hold ($\mathcal{C}$ is a small category with fiber products).

Definition 2.54. *([17], §1 and [26] I, §5.1 and 5.3)*

1. *Let R and U be objects of a category $\mathcal{C}$ and let p and q be two arrows between them. We call $R \underset{q}{\overset{p}{\rightrightarrows}} U$ a categorical equivalence relation in $\mathcal{C}$ of source R and target U if for any object X of $\mathcal{C}$ the induced arrows $h_R(X) \underset{h_q(X)}{\overset{h_p(X)}{\rightrightarrows}} h_U(X)$ define an equivalence relation in the category of sets, this is $h_R(X) \xrightarrow{(h_p(X), h_q(X))} h_U(X) \times h_U(X)$ is a bijection from $h_R(X)$ to the graph of an equivalence relation on $h_U(X)$.*

2. *Let $\mathcal{T}$ be a Grothendieck topology on $\mathcal{C}$. We say that a categorical equivalence relation $R \underset{q}{\overset{p}{\rightrightarrows}} U$ is a $\mathcal{T}-$equivalence relation if the arrows p and q are coverings in $\mathcal{T}$.*

Remark 2.55. The notion of an equivalence relation is at the origin of the notion of groupoid space.

In particular, for the a given scheme S, in the etale site (Sch/S), for any categorical equivalence relation $R \underset{q}{\overset{p}{\rightrightarrows}} U$ there is a unique map $e : U \to R$ such that $p \circ e = q \circ e = \mathrm{id}_U$ (cf. [26], I, §5.2). One can also define the composition map m and the inverse. Any equivalence relation defines in a canonical way a groupoid $S-$space. Thus, by Construction 2.51, we can associate to any equivalence relation $R \underset{q}{\overset{p}{\rightrightarrows}} U$ an $S-$groupoid $[U, R]'$. ♣

Definition 2.56. *1. ([17],§1) Let $R \underset{q}{\overset{p}{\rightrightarrows}} U$ be a categorical equivalence relation in a category $\mathcal{C}$. A pair (Q, π), with Q an object of $\mathcal{C}$ and $\pi : U \to Q$ an arrow in $\mathcal{C}$, is called a categorical quotient for the given categorical equivalence relation if it is a solution of the universal*

$$\begin{tikzcd} R \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & U \arrow[r, "f"] \arrow[d, "\pi"'] & V \\ & Q \arrow[ur, dashed] & \end{tikzcd}$$

Figure 2.8: Categorical quotient for categorical equivalence relation.

problem represented by the diagram of Figure 2.8, where V is any object of $\mathcal{C}$ such that $f \circ p = f \circ q$.

2. *([26], 5.3) Let $R \overset{p}{\underset{q}{\rightrightarrows}} U$ be a $\mathcal{T}-$equivalence relation in a site $\mathcal{C}$ such that*

$$h_X : \mathcal{C} \to (\text{Set}) \textit{ is a sheaf, } \forall X \textit{ object of } \mathcal{C}. \tag{2.2.1}$$

A pair (Q, π), with Q an object of $\mathcal{C}$ and $\pi : U \to Q$, is called a $\mathcal{T}-$quotient of $R \overset{p}{\underset{q}{\rightrightarrows}} U$ if (h_Q, h_π) is a categorical quotient for $h_R \overset{h_p}{\underset{h_q}{\rightrightarrows}} h_U$ in the category of sheaves on $\mathcal{C}$.

Remark 2.57. The categorical quotient defined above is also called a cokernel for the pair (p, q) (cf. [17]) or a co-equalizer (cf. [41]). See [41], Proposition 3.2.4, page 53, for a definition of this notion in terms of inductive limits. ♣

Definition 2.58. 1. *([17], §1 or [26], I, §5.1) We say that a categorical equivalence relation $R \overset{p}{\underset{q}{\rightrightarrows}} U$ is effective if it admits a categorical quotient (Q, π) such that $R = U \times_Q U$.*

2. *([26], I, 5.3) In a site $\mathcal{C}$ such that (2.2.1), we say that a $\mathcal{T}-$equivalence relation $R \overset{p}{\underset{q}{\rightrightarrows}} U$ is effective if it admits a $\mathcal{T}-$quotient (Q, π) such that the morphism $(p, q) : R \to U \times_Q U$ is an isomorphism.*

Remark 2.59. (after [26], I, §5) With the previous notations, let $(\mathcal{C}, \mathcal{T})$ be a site and suppose moreover that (2.2.1) holds. Let $R \overset{p}{\underset{q}{\rightrightarrows}} U$ be a $\mathcal{T}-$equivalence relation. Then, $h_R \overset{h_p}{\underset{h_q}{\rightrightarrows}} h_U$ is a categorical equivalence relation in the category of sheaves on sets on $\mathcal{C}$. There exists a presheaf Q and a natural transformation π satisfying the cokernel universal property for $h_R \overset{h_p}{\underset{h_q}{\rightrightarrows}} h_U$. See Construction 2.60 bellow for an explicit description of the sheaf associated to Q (following [26], I, §5.4).

We have $h_R = h_U \times_Q h_U$. In general, Q is not representable. If Q is representable by an object X of $\mathcal{C}$, we deduce an isomorphism $R \simeq U \times_X U$, that is $R \overset{p}{\underset{q}{\rightrightarrows}} U$ is effective. ♣

Construction 2.60. Let $(\mathcal{C}, \mathcal{T})$ be a site, such that condition (2.2.1) holds. Let $R \rightrightarrows U$ be a $\mathcal{T}-$equivalence relation in $\mathcal{C}$. We denote by E the sheaf h_R, by E' the sheaf h_U and by $E \overset{p}{\underset{q}{\rightrightarrows}} E'$ the categorical equivalence relation induced in the category of sheaves on sets on $\mathcal{C}$ by the previous one. In the category of pre-sheaves, a categorical quotient of $E \overset{p}{\underset{q}{\rightrightarrows}} E'$ exists. In the sequel, we construct the sheaf associated to this categorical quotient. This construction is the incipient idea of the sheafification process (see Section 2.2.3).

Let X be an object of $\mathcal{C}$. We take $\mathcal{P}_X$ to be the set of all pairs $(\{X_i \to X\}, \{a_i\})$ where $\{X_i \to X\}$ is a covering in $\mathcal{T}$ and each a_i is a section in $E'(X_i)$ such that the following condition holds. For any two different indices i and j, we consider the sections $E'(p_i)(a_i)$, respectively $E'(p_j)(a_j)$, images of a_i respectively a_j in $E'(X_i \times_X X_j)$, where p_i and p_j denote the projection on the i^{th}, respectively j^{th} factor (see Figure 3).

$$\begin{tikzcd} X_i \arrow[r] & X \\ X_i \times_X X_j \arrow[u, "p_i"] \arrow[r, "p_j"'] & X_j \arrow[u] \end{tikzcd} \qquad \begin{array}{ccc} a_i & E'(X_i) & \\ \downarrow & \downarrow & \\ E'(p_i)(a_i) & E'(X_i \times_X X_j) & \longleftarrow E'(X_j) \\ & E'(p_j)(a_j) & \longleftarrow\!\shortmid\ a_j \end{array}$$

Figure 2.9: Condition for the pairs of $\mathcal{P}_X$.

We then ask that the pair $(E'(p_i)(a_i), E'(p_j)(a_j))$ is in the image of the arrow

$$E(X_i \times_X X_j) \xrightarrow{(p(X_i \times_X X_j), q(X_i \times_X X_j))} E'(X_i \times_X X_j) \times E'(X_i \times_X X_j). \tag{2.2.2}$$

This means that we require that $E'(p_i)(a_i)$ and $E'(p_j)(a_j)$ are equivalent in $E'(X_i \times_X X_j)$, via the equivalence relation induced by the pair (p, q).

We identify two pairs $(\{X_i \to X\}, \{a_i\})$ and $(\{X'_j \to X\}, \{a'_j\})$ of $\mathcal{P}_X$ if there exists a refinement $\{Y_k \to X\}$ for both $\{X_i \to X\}$ and $\{X'_j \to X\}$ and the following holds. For an arbitrary index k, let $Y_k \to X$ factorize through $X_i \to X$, respectively $X'_j \to X$. We denote by b_{ki}, respectively b'_{kj} the images of a_i, respectively a'_j in $E'(Y_k)$. Then, we ask that the pair (b_{ki}, b'_{kj}) is in the image of the arrow

$$E(Y_k) \xrightarrow{(p(Y_k),q(Y_k))} E'(Y_k) \times E'(Y_k). \tag{2.2.3}$$

In other words, b_{ki} and b'_{kj} are equivalent in $E'(Y_k)$ via the equivalence relation induced by the pair (p, q). One can easily check that the above identification on the pairs of the set $\mathcal{P}_X$ is an equivalence relation $\simeq$. We put $Q^a(X) := \mathcal{P}_X/\simeq$. ♣

Remark 2.61. One can prove that Q^a has a cokernel property similar to the one of Figure 2.8, with V a sheaf on sets on $\mathcal{C}$. The construction of the sheaf Q^a works more generally for an equivalence relation $E \overset{p}{\underset{q}{\rightrightarrows}} E'$ of sheaves on sets on a site. ♣

Remark 2.62. If $E \overset{p}{\underset{q}{\rightrightarrows}} E'$ is a categorical equivalence relation in the category of sheaves on sets on a site, there is no reason why in general the set $Q^a(X)$ defined above should have other peculiar structures.

However, for $E = h_R$, $E' = h_U$ and $R \rightrightarrows U$ a categorical equivalence relation in a site, we can endow $Q^a(X)$ with a groupoid-structure as follows. The set of objects is the set $Q^a(X)$ it-self. To define an arrow between two points P and P' of $Q^a(X)$ it is enough to give arrows between the underlying pairs such that they behave well with respect to the equivalence relation $\simeq$. For this, let P be given by a pair $(\{X_i \to X\}, \{a_i\})$, with a_i in $E'(X_i)$ and P' by $(\{Z_j \to X\}, \{b_j\})$, with b_j in $E'(Z_j)$. Then, an arrow between P and P' is induced by the following commutative diagram of Figure 2.10, where h comes from the universal property of the fiber product, because $R \simeq U \times_{Q^a} U$ (cf. [26], page 74). ♣

$$\begin{tikzcd}
& X_i \times_X Z_j \arrow[dl] \arrow[d, "h"] \arrow[dr] & \\
X_i \arrow[dr] \arrow[ddr, "a_i"'] & R \arrow[dd, bend right=60] \arrow[dd, bend left=60] & Z_j \arrow[dl] \arrow[ddl, "b_j"] \\
& Y \arrow[d] & \\
& U &
\end{tikzcd}$$

Figure 2.10: Commutative diagram for Remark 2.62.

Lemma 2.63. *The presheaf Q^a defined above is a sheaf in the sense of Definition 2.27.*

Proof:

We use mainly the properties of a covering, as stated in Definition 2.15. Let X be any object of $\mathcal{C}$ and $\{a_i : X_i \to X\}$ a $\mathcal{T}-$covering. An element P of $Q^a(X) = \mathcal{P}_X / \simeq$ is the class of a pair $u = (\{Y_j \to X\}, \{b_j\})$, where, for any index j, b_j is a section in $E'(Y_j)$, such that for two indices $j \neq l$ one identifies $E'(p_j)(b_j)$ and $E'(p_l)(b_l)$ via the equivalence relation induced by the pair (p, q). To such a pair, one can associate in $\mathcal{P}_{X_i}$ the pair $u_i := (\{X_i \times_X Y_j \to X_i\}, \{E'(p_j^2)(b_j)\}$, where p_j^2 denotes the projection on the factor Y_j of the fiber product $X_i \times_X Y_j$. We denote by $Qa_i(P)$ the class of u_i in $Q(X_i)$.

Now, for the separateness, with the previous notations consider also a point P' of $Q^a(X)$ given by a pair $v = (\{Z_k \to X\}, \{c_k\})$. We suppose that $Qa_i(P) = Qa_i(P')$, for any index i. This means that there exist a common refinement $\{V_{jk}^i \to X_i\}$ such that the images of $E'(p_j^2)(b_j)$ and $E'(p_k^2)(c_k)$ on V_{jk}^i coincide (as usually via (p, q)). Let d_{jk}^i denote this common value. By the properties of coverings, $\{V_{jk}^i \to X\}_{i,j,k}$ provides a covering for X. Then, by help of $\simeq$, the pair $(\{V_{jk}^i \to X\}_{i,j,k}, \{d_{jk}^i\})$ and u, respectively v define the same point in $Q(X)$. Thus P and P' coincide.

For the effectiveness, with the previous notations, let $u_i = (\{Y_\alpha^i \to X\}, \{b_\alpha^i\})$ be a pair defining a point P_i in $Q(X_i)$ and $u_j = (\{Z_\beta^j \to X\}, \{c_\beta^j\})$ a pair defining a point P_j in $Q(X_j)$. We denote by X_{ij} the fiber product $X_i \times_X X_j$. We suppose that the pairs $(\{Y_\alpha^i \times_{X_i} X_{ij} \to X_{ij}\}, \{E'(p_\alpha^i)(b_\alpha^i)\})$ and $(\{Z_\beta^j \times_{X_j} X_{ij} \to X_{ij}\}, \{E'(p_\beta^j)(c_\beta^j)\})$ define the same point in $Q(X_{ij})$. This means that there is a common refinement $\{V_{\alpha\beta}^{ij}\}_{\alpha,\beta}$ of X_{ij} such that the images of $E'(p_\alpha^i)(b_\alpha^i)$ and $E'(p_\beta^j)(c_\beta^j)$ are the same in $V_{\alpha\beta}^{ij}$ and denote by $e_{\alpha\beta}^{ij}$ this common value. Notice that the elements $e_{\alpha\beta}^{ij}$ and $e_{\alpha\beta}^{ji}$ are the same via this identification. Now, using again the properties of coverings, we recover a pair $(\{V_{\alpha\beta}^{ij} \to X\}_{i,j,\alpha,\beta}, \{e_{\alpha\beta}^{ij}\})$ that provides a point P with the property $Qa_i(P) = P_i$, for any i, as required. ∎

Lemma 2.64. *Let S be a scheme. The etale (respectively fpqc, fppf) site* (Sch/S) *satisfies (2.2.1), of Definition 2.56.*

Proof:

For the statement on the fpqc topology, see [3], Exposé VII, 2 and for a proof see [45], Theorem 2.55. From the relation between topologies (e.g. [27], Chapter 9), this is enough to conclude also on the fppf and etale topologies. ∎

Remark 2.65. Let S be a scheme and $R \overset{p}{\underset{q}{\rightrightarrows}} U$ an etale equivalence relation on the etale site (Sch/S). Consider the induced equivalence relation on sheaves on (Sch/S). Then, the quotient sheaf associated to the equivalence relation $h_R \overset{h_p}{\underset{h_q}{\rightrightarrows}} h_U$ – seen as an $S-$groupoid – is nothing else but the stack associated to the prestack $[U,R]'$ of Construction 2.51. ♣

Lemma 2.66. *Let S be a base scheme. The family of all etale morphisms of the etale site* (Sch/S) *is an effective descent class in the sense of [26], Chapter I, Sections 1.6-9.*

Proof:

We recall that a family $\mathcal{D}$ of arrows of a given site $(\mathcal{C},\mathcal{T})$ (where (2.2.1) of Definition 2.56 holds), satisfies effective descent if the following hold:

1. it is closed, that is it contains all isomorphisms and for any commutative diagram of $\mathcal{C}$
$$\begin{tikzcd} U' \arrow[r] \arrow[d, "f'"'] & U \arrow[d, "f"] \\ X' \arrow[r] & X \end{tikzcd}$$
with f in $\mathcal{D}$, the map f' is also in $\mathcal{D}$;

2. it is stable, this is for any arrow $f : X' \to X$ and any $\mathcal{T}-$covering $\{X_i \to X\}$ of X, if each $f_i : X' \times_X X_i \to X_i$ is in $\mathcal{D}$, then f is in $\mathcal{D}$;

3. if $\{X_i \to X\}$ is a $\mathcal{T}-$covering, F is a sheaf on $\mathcal{C}$ with a map $F \to h_U$ such that for each i the sheaf fiber product $h_{X_i} \times_{h_X} F$ is isomorphic with h_{Y_i} for some object Y_i, then there exists an object Y of $\mathcal{C}$ such that there is an isomorphism $h_Y \to F$ are isomorphic and the corresponding arrow $U \to Y$ is in $\mathcal{D}$.

The first two statements are proved in [26], I.4.11, 5). For 3, for any two indices i and j, we denote by Y_{ij} the fiber product $Y_i \times_X (X_i \times_S X_j)$ and by Y_{ji} the fiber product $Y_j \times_X (X_i \times_S X_j)$. Using the hypothesis on F, X and the given covering, we deduce at functorial level $h_{Y_{ij}} \simeq h_{Y_{ji}}$, thus also an isomorphism $Y_{ij} \simeq Y_{ji}$. This means that we can glue $\{Y_i\}_i$ into a variety Y. Then, F and h_Y coincide locally, so they are the same. ■

Let us see now how to relate the existence of a quotient sheaf with the theory of Deligne-Mumford stacks.

Proposition 2.67. *Let $R \overset{p}{\underset{q}{\rightrightarrows}} U$ be an etale equivalence relation in the etale site* (Sch/S)*. We assume that the map $R \to U \times_S U$ is quasi-compact and separated. Then, the quotient sheaf (Q^a, π) of the induced categorical equivalence relation $h_R \overset{h_p}{\underset{h_q}{\rightrightarrows}} h_U$ has the following* **local representability** *condition*

for any scheme V and any etale map $h_V \to Q$

1. *the fiber product $h_U \times_Q h_V$ is representable;*

2. *$h_U \times_Q h_V \to h_V$ is induced by an etale surjective map of schemes.*

The sheaf Q^a is a Deligne-Mumford stack.

Proof:
The representability of the fiber product $h_U \times_Q h_V$ and the last claim follow from [26], I §5.9, using Lemma 2.66. By Proposition II.1.3, b) op. cit. there exists an algebraic space A – unique up to unique isomorphism – that satisfies the local representability conditions. We conclude that Q^a and A are isomorphic. Moreover, by the same Proposition, the algebraic space A has an etale presentation $h_U \to A$ making it into a Deligne-Mumford stack. Thus the quotient sheaf Q^a is also a Deligne-Mumford stack. ∎

2.2.3 Sheafification of a functor

We recall the sheafification process for functors with values in the category (Set). This theory has its roots in the construction of a quotient sheaf for categorical equivalence relations as in 2.60, though explicit statements were made only in the last years, for example in [45], 2.3.7.

The construction stated below applies for functors with target-category a small category (this is with class of objects a set), such as the category of groups, rings etc. in a given universe. The idea is that all properties of a functor F on a small category $\mathcal{C}$ such as (Ring),(Group) etc. can be recovered from similar properties of the functor induced by composition with the natural forgetful functor on (Set).

Definition 2.68. *([45], Definition 2.63, "sheafification") Let $\mathcal{C}$ be a site and $F : \mathcal{C} \to$ (Set) a contravariant functor. A pair (F^a, T) is called a sheafification of F if:*

1. *F^a is a sheaf, $T : F \to F^a$ is a natural transformation of functors;*

2. *if Y is an object of $\mathcal{C}$, any two sections of $F(Y)$ that coincide on $F^a(Y)$ via T, have the same pull-back on a covering of Y, this is:*
   $\forall Y \in Ob(\mathcal{C})$, $\forall u, v \in F(Y)$ *with* $F^a(Y)(T(Y)(u)) = F^a(Y)(T(Y)(v))$,
   $\exists \{a_i : Y_i \to Y\}$ *with* $F(a_i)(u) = F(a_i)(v), \forall i$;

3. *for any object Y of $\mathcal{C}$ and any element u^a of $F^a(Y)$, there exists a covering $\{a_i : Y_i \to Y\}$ and elements u_i of $F(Y_i)$ such that $F^a(a_i)(u) = T(Y_i)(u_i)$, for any i.*

The main result on the topic is the following:

Theorem 2.69. *Let $\mathcal{C}$ be a site and F a functor on $\mathcal{C}$. Then there exists a sheafification F^a which is unique up to canonical isomorphism and such that the following universal property holds: for any sheaf F', any arrow $F \to F'$ factors uniquely through F^a.*

Proof:

The proof, following [45], Proposition 2.64, uses a two-step construction. First, associate to F a functor F^s having the separateness property of Definition 2.27. We say that for an object Y, two elements u and v of $F(Y)$ are equivalent if they coincide on a cover of Y. This defines an equivalence relation $\equiv$ on the set $F(Y)$. We put $F^s(Y) := F(Y)/\equiv$. If $Y \to Z$ is an arrow, we also get $F^sZ \to F^sY$. We have a natural transformation $F \to F^s$.

We want to associate to F^s a sheaf. The construction goes as the one of 2.60. For an object Y consider the set $\mathcal{P}_Y$ formed by the pairs $(\{Y_i \to Y\}, \{a_i\})$, where $\{Y_i \to Y\}$ is a covering in $\mathcal{C}$ and each a_i is a section in $F^s(Y_i)$. We ask that for any two indices i and j, the pull-back of a_i and a_j to $F^s(Y_i \times_Y Y_j)$, along the first and second projection respectively, coincide. This means that in (2.2.2), we consider the particular equivalence relation defined by the diagonal.

On the set $\mathcal{P}_Y$, we identify two pairs $(\{Y_i \to Y\}, \{a_i\})$ and $(\{Y_i' \to Y\}, \{a_i'\})$ as in (2.2.3). This time we take the common refinement to be the one provided by the fiber product $\{Y_i \times_Y Y_j' \to Y\}$ and we ask that the restrictions of a_i and a_j' along the first and the second projection are equal (in other words, we consider again the equivalence relation induced by the diagonal on the set $E(Y)$).

We denote by $\simeq$ the equivalence relation thus defined on $\mathcal{P}_Y$. The transitivity of this relation follows from the separateness of the functor F^s. We define $F^a(Y) := \mathcal{P}_Y/\simeq$. By construction, there is a natural transform $F^s \to F^a$, thus also $F \to F^a$. Cf. [45] for the universal property. ∎

Remark 2.70. We use here in an essential way the fact that for any object Y of $\mathcal{C}$, the target $F(Y)$, thus also $F^s(Y)$, is a set. Because of this, we can consider sections in the set $F(Y)$. This construction can not be performed in general for $2-$functors, but a similar one holds by help of $2-$colimits. ♣

Remark 2.71. We use here the remark on page 46 of [45] to give a different approach on the sheafification process. We remark that the construction of a sheafification of a functor F as in 2.69 (or the construction of a quotient for a categorical equivalence relation as in 2.60) uses mainly the existence of a common refinement for two given ones in a site. Thus, we can formalize the above definition of F^a by help of direct limits. We recall (see Remark 2.16 or [45], 2.38) that a sieve on an object Y of $\mathcal{C}$ is a sub-functor of h_Y or – which is the same – a collection of arrows $\mathcal{S}(Y) := \{T \to Y\}$ such that every composition $T' \to T \to Y$ is still in $\mathcal{S}(Y)$. The family of all sieves on a given object Y is an ordered family: we say that $\mathcal{S} \le \mathcal{S}'$ if $\mathcal{S}'$ is a subset of $\mathcal{S}$. By [45], Proposition 2.44, the family of all sieves on Y forms a direct system, so that we can take the direct limit of a system indexed by the set of all sieves. Then, for an object Y, we have a canonical bijective correspondence:

$$F^a(Y) = \varinjlim_{\substack{\mathcal{S} \text{ sieve} \\ \text{on } Y}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{S}, F^s) \tag{2.2.4}$$

Here, the notation $\mathrm{Hom}_{\mathcal{C}}$ means that we consider the functors $\mathcal{S}$, as sub-functor of h_Y and F^s, both defined on the same site $\mathcal{C}$. ♣

2.3 Stacks on $\mathfrak{Cat}$

We consider here functors defined on a site $(\mathcal{C}, \mathcal{T})$ with values in the category of all categories $\mathfrak{Cat}$. As usually the category $\mathcal{C}$ has fiber products. We want to see when such a functor is a stack. A possible way to see this is by help of the associated fibered category, as done in Construction 2.75. This is far too general to be practical. Another construction can be achieved by imitating Definition 2.32, using an accessory functor Hom and asking for a separateness condition. This definition is very useful for the case when the source-site $\mathcal{C}$ is the site associated to a topological space, but it is not convenient for the other cases.

One of the main questions is how to perform a sheafification (official term: stackification) process to associate a stack to a $2-$functor. For some particular cases (e.g. strict $2-$functors, see bellow), we suggest a similar construction similar to the one given in 2.60 or in the proof of Theorem 2.69 (see Section 2.4).

2.3.1 $2-$functors

We recall here some definitions and notations.

Definition 2.72. *1. ([45], Definition 3.10 , [46], §A.1.1; "[lax] $2-$functor/pseudo-functor") Let $F : \mathcal{C} \to \mathfrak{Cat}$ be an arrow from a category $\mathcal{C}$ to the category of all categories, such that $F(Y)$ is a small category for any object Y of $\mathcal{C}$. We say that F is a contravariant [lax] $2-$functor or a pseudo-functor if:*

(a) for each arrow $s : Y \to Y'$ in $\mathcal{C}$ there exists a functor of categories $F(s) : F(Y') \to F(Y)$,

(b) there is a natural transformation $\Psi_Y : F(id_Y) \tilde{\to} id_{F(Y)}$ which is an isomorphism,

(c) for any two composable arrows s, s' of $\mathcal{C}$ there is a natural transformation $\Psi_{s,s'} : F(s' \circ s) \tilde{\to} F(s) \circ F(s')$ which is an isomorphism such that:

i. for any morphism $s : Y \to Y'$ we have $(\Psi_Y \bullet F(s)) \circ \Psi_{id_Y, s} = (F(s) \bullet \Psi_{Y'}) \circ \Psi_{s, id_{Y'}} = id_{F(Y')}$,

ii. *for any three arrows* $Y \xrightarrow{s} Y' \xrightarrow{s'} Y'' \xrightarrow{s''} Y'''$ *we have* $(\Psi_{s,s'} \bullet F(s'')) \circ \Psi_{s \circ s', s''} = (F(s) \bullet \Psi_{s',s''}) \circ \Psi_{s' \circ s'', s}$.

2. *If* Ψ_Y *is the identity for any object* Y *of* $\mathcal{C}$, *we call* F *a* $2-$*functor with strict identities or a prestack.*

3. *If* F *is a prestack such that moreover for all pairs of arrows* (s, s') *the functor* $\Psi_{s,s'}$ *is the identity, we call* F *a strict* $2-$*functor.*

Notation 2.73. For a $2-$functor F we sometimes call the category $F(U)$ the fiber of F over U. From the point of view of Construction 2.75 this name is correct. ♣

Remark 2.74. A strict $2-$functor is just a contravariant functor in the usual sense, from the category $\mathcal{C}$ to the underlying $1-$category of $\mathfrak{Cat}$. ♣

Construction 2.75. (fibered category associated to a $2-$functor) To a pseudo-functor $F : \mathcal{C} \to \mathfrak{Cat}$, one can associate a fibered category on $\mathcal{C}$. We give here only the construction, for a proof see [45], Proposition 3.1.3. Let $\mathcal{F}$ denote the fibered category associated to F. For an object Y of $\mathcal{C}$, the fiber $\mathcal{F}(Y)$ is the category $F(Y)$. The objects of $\mathcal{F}$ are the pairs (Y, u), with Y an object of $\mathcal{C}$ and u an object of $F(Y)$. An arrow $(s, f) : (Y, u) \to (Y', u')$ consists of an arrow $s : Y \to Y'$ in $\mathcal{C}$ and of an arrow $f : F(s)(u') \to u$. The composition of such two arrows $(s, f) : (Y, u) \to (Y', u')$ and $(s', f') : (Y', u') \to (Y'', u'')$ is given by the pair $(f \circ F(s)(f') \circ \Psi_{s,s'}(u''), s' \circ s)$.

If F has values in (Groupoids), then the associated fibered category is a category fibered in groupoids as in Definition 2.12, (2). ♣

Remark 2.76. In general, the fibered category associated to a $2-$functor is not a stack. As seen in the proof of Lemma 2.24, (2), for a fibered category there is a natural way to associate a functor from the fiber-category to the descent category. Then, we say that a $2-$functor is a stack if its associated fibered category is a stack in the sense of Definition 2.25. ♣

Let us see an analogous way to define a stack while starting with a $2-$functor.

Notation 2.77. If $(\mathcal{C}, \mathcal{T})$ is a site and Y is an object of $\mathcal{C}$, we denote by $\mathcal{T}(Y)$ the category of objects Z "over" Y, that is objects Z together with a $\mathcal{T}-$ covering $s_Z : Z \to Y$. ♣

Definition 2.78. 1. *Let* F *be a* $2-$*functor on* $\mathcal{C}$. *We say that* F *is separated if for any object* Y *of* $\mathcal{C}$ *any two objects* u *and* v *of* $F(Y)$ *the contravariant functor:*

$$\begin{cases} \mathcal{T}(Y) & \to & \text{(Set)} \\ Z & \mapsto & \mathrm{Hom}_{F(Z)}(F(s_Z)(u), F(s_Z)(v)) \end{cases}$$

is a sheaf.

2. *A separable $2-$functor is a stack if every descent data is effective in the sense of Definition 2.32, (2).*

2.4 Special Deligne-Mumford stacks

In the literature, the main interest is to link the theory of stacks and sheaves with practical notions from algebraic geometry, number theory etc. The "algorithm" to follow is:

– state you problem in terms of categories
– get a link with the theory of stacks
– solve your problem there
– try to go back and find the answer for your case.

Thus, it is interesting to have examples and constructions that can provide stacks. Among the most useful stacks introduced to solve problems from algebraic geometry or number theory are the algebraic stacks and the Deligne-Mumford stacks of Section 2.1.3.

In the sequel, following [22], we introduce the notion of a Deligne-Mumford stack associated to a pair (X, D), with X normal variety and D an effective $\mathbb{Q}-$divisor fixed on it. The aim is to consider the derived category of sheaves on this stack instead of the category of coherent sheaves on the variety X.

2.4.1 The framework

All along the section we fix a base scheme S, noetherian and separated. We consider the etale site $\mathcal{C} := (\mathrm{Sch}/S)$, as recalled in Example 2.19. We denote by $\mathcal{E}$ the etale topology on this site. A scheme over S means a scheme of finite type over S.

In the sequel, X denotes a normal scheme and D an effective $\mathbb{Q}-$divisor on X with coefficients in the set $\{1 - 1/n \mid n \in \mathbb{N}^*\}$. We assume that the following condition holds:

> There exists a quasi-finite, surjective morphism, $\pi : U \to X$, with U smooth variety and such that:
>
> $$\pi^*(K_X + D) = K_U \tag{2.4.1}$$

We denote by R the normalization of the fiber product $U \times_X U$ and by p (respectively q) the first (respectively the second) projection from R to U.

Remark 2.79. Condition (2.4.1) holds for any finite quotient, that is X of the form M/G, for M smooth variety and G finite group acting faithfully on it. The divisor D is a measure to quantify the ramification (in codim 1). The case $D = 0$ is important and non-trivial and it can occur precisely when there is no ramification, that is when the group G acts without pseudo-reflections.

A particular case when one can consider $D = 0$ is the case $M = \mathbb{A}^n$ and G finite subgroup of $SL_n(\mathbb{C})$. ♣

Proposition 2.80. *The diagram* $R \overset{p}{\underset{q}{\rightrightarrows}} U$ *defines an etale equivalence relation in the category of schemes.*

Proof:

We want to prove that the morphism p is etale. A similar proof works for q.

We recall ([18], Section II.3, Exercise 3.8) that the normalization morphism $\eta : R \to U \times_X U$ is a finite morphism, because of the assumption that every scheme is of finite type over S. Let us denote by $\mathrm{Sm}(R)$ the inverse image under η of the non-singular locus $U \times_X U \setminus \mathrm{Sing}(U \times_X U)$. We have an isomorphism $\eta_{|\mathrm{Sm}(R)} : \mathrm{Sm}(R) \to U \times_X U \setminus \mathrm{Sing}(U \times_X U)$. In the diagram of Figure 2.11 we have the situation up to now.

$$\begin{tikzcd}
\begin{matrix} R \\ \text{normal} \end{matrix} \arrow[dr, "\eta", "\text{finite}"'] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & \\
& U \times_X U \arrow[r, "p_1"] \arrow[d, "p_2"] & U_{\text{smooth}} \arrow[d, "\pi"', "\text{quasi-finite}"] \\
& U \arrow[r, "\pi"'] & X
\end{tikzcd}$$

Figure 2.11: Relation between X, U and R.

Now, by (2.4.1), we have $\pi^*(K_X + D) = K_U$, which implies in particular that the codimension of the support of $\Omega_{U/X}$ is greater than 2 (there is no divisor on the ramification locus). The support of $\Omega_{U/X}$ and the ramification locus of π are the same.

We prove that $\mathrm{codim}(\mathrm{Ram}(\pi)) \geq 2$. We take the variety $U \setminus \mathrm{Ram}(\pi)$, which is smooth. Let us denote by X_{nr} the set $\pi(U \setminus \mathrm{Ram}(\pi))$. Now, the restriction $\pi_{|U \setminus \mathrm{Ram}(R)}$ is etale and we have a commutative diagram as in Figure 2.12.

Because the base extension of an etale morphism is also etale, we deduce that $(p_1)_{|(\pi \circ p_1)^{-1}(X_{\mathrm{nr}})}$ is etale. We deduce that $(\pi \circ p_1)^{-1}(X_{\mathrm{nr}})$ is smooth because $U \setminus \mathrm{Ram}(\pi)$ is. In particular, $(\pi \circ p_1)^{-1}(X_{\mathrm{nr}})$ coincide with its normalization, so we get an isomorphism:

$$\eta^{-1}((\pi \circ p_1)^{-1}(X_{\mathrm{nr}})) \simeq (\pi \circ p_1)^{-1}(X_{\mathrm{nr}}).$$

We deduce the diagram of Figure 2.13, where the restriction $p_{|\eta^{-1}((\pi \circ p_1)^{-1}(X_{\mathrm{nr}}))}$ is etale as composition of an isomorphism and of the etale map $(p_1)_{|(\pi \circ p_1)^{-1}(X_{\mathrm{nr}})}$.

$$\begin{tikzcd}
(\pi \circ p_1)^{-1}(X_{\mathrm{nr}}) \arrow[r, "(p_1)_{|(\pi\circ p_1)^{-1}(X_{\mathrm{nr}})}"] \arrow[d, "(p_1)_{|(\pi\circ p_1)^{-1}(X_{\mathrm{nr}})}"'] & U \setminus \mathrm{Ram}(\pi) \arrow[d, "\pi_{|U\setminus \mathrm{Ram}(R)}"] \\
U \setminus \mathrm{Ram}(\pi) \arrow[r, "\pi_{|U\setminus \mathrm{Ram}(R)}"'] & X_{\mathrm{nr}}
\end{tikzcd}$$

Figure 2.12: Etale morphisms for Lemma 2.80.

$$\begin{tikzcd}
\eta^{-1}((\pi \circ p_1)^{-1}(X_{\mathrm{nr}})) \arrow[rr, "\sim"] \arrow[dr, "p_{|\eta^{-1}((\pi\circ p_1)^{-1}(X_{\mathrm{nr}}))}"'] & & (\pi \circ p_1)^{-1}(X_{\mathrm{nr}}) \arrow[dl, "(p_1)_{|(\pi\circ p_1)^{-1}(X_{\mathrm{nr}})}"] \\
& U \setminus \mathrm{Ram}(\pi) &
\end{tikzcd}$$

Figure 2.13: Commutative diagram for Lemma 2.80

This implies that $p(\mathrm{Ram}p) \subset \mathrm{Ram}\pi$. By the purity theorem, we conclude that $\mathrm{codim}(\mathrm{Ram}(\pi)) \geq 2$ (else $\mathrm{Ram}p =$). We conclude that $R \xrightarrow{p} U$ is etale. ∎

Proposition 2.81. *Let U and U' be two smooth varieties and $\alpha : U' \to U$ a quasi-finite, dominant morphism. Then, α is etale if and only if it is crepant.*

Proof:

A quasi-finite, dominant morphism is finite on an open dense subset (see [18], page 90). In particular, we deduce that the field of fractions of U and U' have the same transcendence degree, so $\dim U = \dim U'$. We denote by d this dimension.

For the morphism $\alpha : U' \to U$ we deduce ([18], II, Proposition 8.11) the existence of an exact sequence of sheaves:

$$\alpha^*\Omega_U \xrightarrow{h} \Omega_{U'} \longrightarrow \Omega_{U'/U}. \tag{2.4.2}$$

Suppose that α is etale. Then, by Remark 2.4, 5), $\Omega_{U'/U} = 0$, so the above exact sequence becomes

$$\alpha^*\Omega_U \longrightarrow \Omega_{U'} \longrightarrow 0\ .$$

Now, both Ω_U and $\Omega_{U'}$ are locally free of same dimension d, so we deduce an isomorphism $\alpha^*\Omega_U \simeq \Omega_{U'}$. We conclude that $\alpha^*(\Omega_U^{\otimes d}) \simeq \Omega_{U'}^{\otimes d}$. Considering the action of the symmetric group, we deduce $\alpha^*(\wedge^d \Omega_U) = \wedge^d \Omega_{U'}$, that is α is crepant.

Suppose now that α is crepant. We argue by contradiction and suppose that $\Omega_{U'/U}$ is not zero. Then, there exists a point x such that $(\Omega_{U'/U})_x$ is not zero and by (2.4.2) we have an exact sequence

$$(\alpha^*\Omega_U)_x \xrightarrow{h_x} (\Omega_{U'})_x \longrightarrow (\Omega_{U'/U})_x.$$

On the other hand, we know that α is crepant, so $\det h_x$ is invertible, thus in particular h_x is an isomorphism for any y. This gives a contradiction. ∎

2.4.2 The construction

We want to associate to the pair (X, D) an $S-$groupoid, such that the result is a Deligne-Mumford stack. The idea is as follows. We fix a scheme U such that (2.4.1) holds. We associate to U a $2-$functor on groupoids F_U. In general, this functor is not a stack. We describe explicitly the sheafification process of F_U in 2.4.2.2. The resulting sheaf F_U^a is actually a Deligne-Mumford stack. We then prove that the result doesn't depend on the choice of the etale covering U, but only on the pair (X, D). We call it the smooth Deligne-Mumford stack associated to the pair (X, D).

2.4.2.1 $2-$functor in groupoids associated to the pair $(\mathbf{X}, \mathbf{D})$

We define an arrow from the category of schemes over S to the category of groupoids:

$$\begin{array}{rcl} F_U : (\mathrm{Sch}/S) & \rightarrow & (\mathrm{Groupoid}) \\ Y & \mapsto & F_U(Y) \end{array}$$

For an $S-$scheme Y, we want to define the category $F_U(Y)$. We put the set of objects of the category $F_U(Y)$ to be the set of arrows $h_U(Y) = \mathrm{Hom}_{(\mathrm{Sch}/S)}(Y, U)$.

To define the set of arrows in the category $F_U(Y)$, let $u : Y \to U$ and $v : Y \to U$ be two arrows in $h_U(Y)$. A morphism between u and v in the groupoid $F_U(Y)$ is an arrow $f : Y \to R$ such that we have a commutative diagram as in Figure 2.14.

To define the composition map, first we remark that there is a morphism

$$R \times_U R \longrightarrow (U \times_X U) \times_U (U \times_X U) = U \times_X U \times_X U \xrightarrow{p_{12}} U \times_X U,$$

where p_{12} denotes the projection on the first and on the second factor. The scheme $R \times_U R$ is normal, by Proposition 2.80 above, so by the universal property of the normalization it follows that we have a factorization:

$$\begin{tikzcd} R \times_U R \arrow[r] \arrow[dr, dashed, "\pi_{12}"'] & U \times_X U \\ & R \arrow[u] \end{tikzcd}$$

$$
\begin{tikzcd}
& Y \arrow[d, "f"] \arrow[ddl, "u"'] \arrow[ddr, "v"] & \\
& R \arrow[dl, "p"] \arrow[dr, "q"'] & \\
U \arrow[dr, "\pi"'] & & U \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}
$$

Figure 2.14: Arrow in the groupoid $F_U(Y)$.

We also denote by π_{23} (respectively π_{13}) the corresponding maps where we project on the second and third factor (respectively on the first and on the third).

Let now u, v and w be three objects of $F_U(Y)$ and let f be a morphism between u and v and g a morphism between v and w. Then, the composition $g \circ f$ is a same as to give an arrow $h : Y \to R \times_U R$ such that the composition $Y \xrightarrow{h} R \times_U R \xrightarrow{\pi_{13}} R$ makes the diagram of Figure 2.15 commutative (here, the dotted arrows are the one related with the composition map).

$$
\begin{tikzcd}
& & Y \arrow[d, dashed, "\mathrm{h}"] \arrow[ddl, bend right, "f"'] \arrow[dddll, bend right, "u"'] \arrow[ddd, bend right, "v"'] \arrow[ddr, bend left, "g"] \arrow[dddrr, bend left, "w"] & & \\
& & R \times_U R \arrow[dl, "\pi_{12}"'] \arrow[d, dashed, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & & \\
& R \arrow[dl] \arrow[dr] & R \arrow[dll, dashed, "p"] \arrow[drr, dashed, "q"'] & R \arrow[dl] \arrow[dr] & \\
U & & U & & U
\end{tikzcd}
$$

Figure 2.15: Composition of arrows in the category $F_U(Y)$.

We remark that $F_U(Y)$ thus defined is a groupoid, in which the inverse is given by switching p and q.

For any arrow $Y \xrightarrow{s} Y'$ we define a functor $F_U(s)$ from $F_U(Y')$ to $F_U(Y)$, given by the composition on the right with s, as in Figure 2.16.

$$
\begin{tikzcd}
Y \arrow[d, "s"] \arrow[dd, bend right, "u'\circ s"'] \\
Y' \arrow[d, "u'"] \\
U
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & Y \arrow[d, "s"] \arrow[dd, bend right, "f'\circ s"'] \arrow[dddll, bend right, "u'\circ s"'] \arrow[dddrr, bend left, "v'\circ s"] & & \\
 & & Y' \arrow[d, "f'"] \arrow[ddll, "u'"'] \arrow[ddrr, "v'"] & & \\
 & & R \arrow[dll, "p"] \arrow[drr, "q"'] & & \\
U \arrow[drr, "\pi"'] & & & & U \arrow[dll, "\pi"] \\
 & & X & &
\end{tikzcd}
$$

Figure 2.16: Natural transform $F_U(s)$ on objects and arrows.

For the identity map $Y \xrightarrow{id_Y} Y$, the functor $F_U(id_Y)$ is nothing else but the identity functor of the category $F_U(Y)$. For any two morphisms s and s' in the category (Set/S), $Y \xrightarrow{s} Y' \xrightarrow{s'} Y''$, we have an equality of functors $F_U(s' \circ s) = F_U(s') \circ F_U(s)$.

Lemma 2.82. *The arrow F_U defined above is a strict $2-$functor.*

Proof:

Using the notations of Definition 2.72, (1), for any $S-$scheme Y, the natural transformation Ψ_Y is the identity and for any pair of arrows (s, s') the natural transformation $\Psi_{s,s'}$ is the identity. Thus, conditions 1(c)i and 1(c)ii of the same definition automatically hold. ∎

2.4.2.2 Sheafification of $\mathbf{F_U}$

The aim in what follows is to construct the sheafification for the functor F_U defined above.

Proposition 2.83. *The $2-$functor F_U defined in 2.4.2.1 admits a sheafification $F_U^a : \mathcal{C} \to (\mathrm{Groupoid})$.*

Proof:

We follow a similar construction as the one in the proof of Theorem 2.69 and in the subsequent Remark 2.71. For a scheme Y, we define a category $\mathcal{P}_Y$. We define $F_U^a(Y)$ to be the category $\mathcal{P}_Y$ where we "inverse all the arrows". Let us see this construction.

Let Y be a scheme. We define $\mathcal{P}_Y$ to be the set formed with pairs $(\{\ Y_i \xrightarrow{f_i} Y\ \}, \{u_i\})$, where $Y_i \xrightarrow{f_i} Y$ is an etale covering and u_i is an arrow of $\mathrm{Hom}_{\mathcal{C}}(Y_i, U)$, with a commutative diagram as in Figure 2.17.

$$\begin{tikzcd}
& Y_i \times_Y Y_j \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\
Y_i \arrow[dr, "f_i"] \arrow[ddr, "u_i"'] & & Y_j \arrow[dl, "f_j"'] \arrow[ddl, "u_j"] \\
& Y & \\
& U &
\end{tikzcd}$$

Figure 2.17: Commutative diagram for the definition of $\mathcal{P}_Y$.

Here the notation p_i means that we take the projection on the factor indexed by i. A pair in this set is denoted by capital letters P, Q, while the first term of such a pair – this is a covering of Y – is denoted by $\mathbb{Y}$ and the second term is denoted by $\{u\}$.

We put a structure of a category on the set $\mathcal{P}_Y$ as follows. First we define the set of arrows. Let $P = (\{\ Y_i \xrightarrow{f_i} Y\ \}, \{u_i\})$ and $Q = (\{\ Z_a \xrightarrow{g_a} Y\ \}, \{v_a\})$ be two elements in $\mathcal{P}_Y$. An arrow between Q and P is a pair $\{i(a), h_a\}$, for indices $i(a)$ and maps h_a such that:

1. $\{\ Z_a \xrightarrow{g_a} Y\ \}$ is a refinement of $\{\ Y_i \xrightarrow{f_i} Y\ \}$ that is (see Definition 2.17):
$$\forall a, \exists i(a), \exists\ Z_a \xrightarrow[\text{etale}]{h_a} Y_{i(a)} \ \text{ with } g_a = f_{i(a)} \circ h_a$$

2. with the above notations, there is a compatibility with the arrows $\{u\}$ and $\{v\}$:
$$v_a = u_{i(a)} \circ h_a.$$

We remark that under these conditions, for two indices a and b, by the universal property of the fiber product $Z_a \times_Y Z_b$ we have $g_a \circ p_a = g_b \circ p_b$, from which we deduce $f_{i(a)} \circ (h_a \circ p_a) = f_{i(b)} \circ (h_b \circ p_b)$. Thus, by universal property for the fiber product $Y_{i(a)} \times_Y Y_{i(b)}$, we deduce the existence of an unique arrow $p : Z_a \times_Y Z_b \to Y_{i(a)} \times_Y Y_{i(b)}$ such that the diagram of Figure 2.18 is commutative. In particular, we also deduce that $h_a \circ p_a = p_{i(a)} \circ p, h_b \circ p_b = p_{i(b)} \circ p$. These remarks allow us to define the composition of two arrows in a natural way.

$$\begin{tikzcd}
 & & Z_a \times_Y Z_b \arrow[dll, "p_a"'] \arrow[d, dashed, "p"] \arrow[drr, "p_b"] & & \\
Z_a \arrow[dr, "h_a"] \arrow[ddrr, bend right, "g_a"'] \arrow[dddrr, bend right=60, "v_a"'] & & Y_{i(a)} \times_Y Y_{i(b)} \arrow[dl, "p_{i(a)}"'] \arrow[dr, "p_{i(b)}"] & & Z_b \arrow[dl, "h_b"'] \arrow[ddll, bend left, "g_b"] \arrow[dddll, bend left=60, "v_b"] \\
 & Y_{i(a)} \arrow[dr, "f_{i(a)}"] \arrow[ddr, "u_{i(a)}"'] & & Y_{i(b)} \arrow[dl, "f_{i(b)}"'] \arrow[ddl, "u_{i(b)}"] & \\
 & & Y & & \\
 & & U & &
\end{tikzcd}$$

Figure 2.18: Commutative diagram for definition of an arrow in $\mathcal{P}_Y$.

Thus, the set $\mathcal{P}_Y$ becomes a category (the objects are the elements of the set and we put an arrow between two elements P and Q as described above).

Now, we take the category $\mathcal{P}_Y$ and inverse all its arrows, that is we consider the category of all isomorphism classes of objects of $\mathcal{P}_Y$. This defines a groupoid which we denote by $\mathcal{Q}_Y$. We put $F_U^a(Y) = \mathcal{Q}_Y$.

We prove that the arrow F_U^a thus defined is a functor. First, we prove that if $Y \to Z$ is a morphism of schemes, then we have a functor between and $\mathcal{P}_Z$ and $\mathcal{P}_Y$. On the objects, such a functor is defined as follows. Let $(\mathbb{Z}, \{v\}) := (\{\, Z_a \xrightarrow{g_a} Y \,\}, \{v_a\})$ be an object in the category $\mathcal{P}_Z$. We associate to it an object $(\mathbb{Y}, \{u\})$ as follows. We identify the fiber product $Z \times_Z Y$ with Y. Then the covering $\mathbb{Y}$ is defined by $f_a := g_a \times \mathrm{id} : Z_a \times_Z Y \to Y$. This is an etale covering of Y because base-change preserves etale morphisms. The morphisms u_a are defined by the composition:

$$Z_a \times_Z Y \xrightarrow{p_a} Z_a v_a \longrightarrow U \,.$$

Here, the notation p_a stands for the projection on the factor Z_a of the fiber product $Z_a \times_Z Y$. The map thus defined on objects can be extended in a natural way to arrows.

Taking isomorphism classes, we recover a functor between $\mathcal{Q}_Z$ and $\mathcal{Q}_Y$. This shows that F_U^a is well defined. By the proof of Theorem 2.69, the functor F_U^a is the sheafification of the functor F_U. ■

Notation 2.84. We denote by $\mathcal{F}_U$ the fibered category associated to F_U^a (see Construction 2.75). ♣

Corollary 2.85. *The fibered category $\mathcal{F}_U$ is a Deligne-Mumford stack, with $U \to \mathcal{F}_U$ as etale presentation.*

Proof:

By Lemma 2.80, $R \overset{p}{\underset{q}{\rightrightarrows}} U$ defines an etale equivalence relation, so we can apply the theory of Section 2.2.2. Let Q_U^a denote the quotient sheaf associated to the etale equivalence relation $R \overset{p}{\underset{q}{\rightrightarrows}} U$, as constructed in 2.60. Following Remark 2.62 and Lemma 2.63, we deduce that Q_U^a is a sheaf on groupoids on the etale site (Sch/S). Moreover, by Proposition 2.67 its associated stack is a Deligne-Mumford stack. On the other hand, by Constructions 2.31 and 2.75, and using the descriptions of Q_U^a (Construction 2.60) and respectively of the sheafification process for F_U as described in Section 2.4.2.2, we conclude that $\mathcal{F}_U$ and the stack associated to the sheaf Q_U^a are the same. Thus, $\mathcal{F}_U$ is also a Deligne-Mumford stack. ■

2.4.2.3 Independence

The only thing to prove is that the above sheafification doesn't depend on the choice of the pair (U, π) as in (2.4.1), but only on (X, D).

Proposition 2.86. *Let (U, π) be a pair such that (2.4.1) and $\alpha : U' \to U$ a morphism of finite type, which is an etale covering of U. Then, for any scheme Y, one has an equivalence of categories $\mathcal{F}_U(Y) \simeq \mathcal{F}_{U'}(Y)$.*

Proof:

Let π' denote the composition $\pi \circ \alpha$. Because α is etale, U' is also smooth.

$$\begin{tikzcd} & U' \arrow[dl, "\alpha"'] \arrow[dr, "\pi'"] & \\ U \arrow[rr, "\pi"'] & & X \end{tikzcd}$$

By Remark 2.4, α is quasi-finite. We apply Proposition 2.81 and see that the pair (U', π') also satisfies (2.4.1). We have:

$$(\pi')^*(K_X + D) = \alpha^*(\pi^*(K_X + D)) = \alpha^* K_U = K_{U'}.$$

We prove that for any scheme Y, the categories $F^a_{U'}(Y)$ and $F^a_U(Y)$ are equivalent. Following the sheafification process of Lemma 2.83, for the definition of $F^a_U(Y)$, we considered the category $\mathcal{P}_Y$ whose objects are the pairs $(\{\, Y_i \xrightarrow{f_i} Y \,\}, \{u_i\})$, with $\{\, Y_i \xrightarrow{f_i} Y \,\}$ etale covering and u_i is an arrow in $\mathrm{Hom}_{\mathcal{C}}(Y_i, U)$, such that a commutative diagram as in Figure 2.17 holds. Let us denote $\mathcal{P}_Y$ by $\mathcal{P}$. For the functor $F^a_{U'}(Y)$, we consider the corresponding category $\mathcal{P}'_Y = \mathcal{P}'$ whose objects are the pairs $(\{\, Y'_i \xrightarrow{f'_i} Y \,\}, \{u'_i\})$ with similar conditions. We recall that $F^a_U(Y)$ is then defined as the groupoid $\mathcal{Q}$ obtained from $\mathcal{P}$ by inverting all maps.

We define two functors M and N between the categories $\mathcal{P}$ and $\mathcal{P}'$:

$$\mathcal{P}' \underset{N}{\overset{M}{\rightleftarrows}} \mathcal{P} .$$

We prove that they have "good properties" that allow us to conclude that the corresponding functors

$$\mathcal{Q}' \underset{\mathfrak{N}}{\overset{\mathfrak{M}}{\rightleftarrows}} \mathcal{Q}$$

are inverse to each other. Thus the equivalence between $F^a_U(Y) := \mathcal{Q}$ and $F^a_{U'}(Y) := \mathcal{Q}'$.

In the sequel, we define the functors N, respectively M on the objects. The definition of those functors on the arrows follow naturally and we do not give it.

For the definition of N, let us consider an object of $\mathcal{P}$, that is a pair $(\{\, Y_i \xrightarrow{f_i} Y \,\}, \{u_i\})$, with $\{\, Y_i \xrightarrow{f_i} Y \,\}$ etale covering and $u_i \in \mathrm{Hom}_{\mathcal{C}}(Y_i, U)$. Because $U' \to U$ is etale, by base-change, the second projection $p_i : U' \times_U Y_i \to Y_i$ is also an etale map. By composing it with the etale map $Y_i \to Y$, we conclude get an object of the category $\mathcal{P}'$

$$(\{\, U' \times_U Y_i \xrightarrow{f_i \circ p_i} Y \,\}, \{u'_i\})$$

where u'_i is the first projection from the fiber product $U' \times_U Y_i$ on U'. See also Figure 2.19.

For the definition of M, we consider an object $(\{\, Y'_i \xrightarrow{f'_i} Y \,\}, \{u'_i\})$ in $\mathcal{P}'$ and we associate to it the object $(\{\, Y'_i \xrightarrow{f'_i} Y \,\}, \{\alpha \circ u'_i\})$.

We first remark that the functors M and N are both faithful. We give the proof for N in the sequel, while the proof for M is similar. If $P := (\mathbb{Y}, \{u\})$ and $Q := (\mathbb{Z}, \{v\})$ are two points of $\mathcal{P}$, we want to prove that the map $\mathrm{Hom}_{\mathcal{P}}(P, Q) \to \mathrm{Hom}_{\mathcal{P}'}(NP, NQ)$ is injective. We denote by $\mathrm{Hom}(\mathbb{Y}, \mathbb{Z})$ the set of homomorphism between etale coverings of Y. We remark that $\mathrm{Hom}_{\mathcal{P}}(P, Q)$ is a subset of $\mathrm{Hom}(\mathbb{Y}, \mathbb{Z})$ and $\mathrm{Hom}_{\mathcal{P}'}(NP, NQ)$ is a subset of

$$
\begin{tikzcd}
 & Y_i \arrow[dl, "u_i"'] \arrow[dr, "f_i"] & & & U' \times_U Y_i \arrow[dl] \arrow[dr, "p_i"] & & \\
U & & Y \arrow[r, mapsto, dotted] & U' \arrow[dr, "\alpha"] & & Y_i \arrow[dl, "f_i"] \arrow[dr, "u_i"] & \\
 & & & & U & & Y
\end{tikzcd}
$$

Figure 2.19: Definition of the functor N.

$\mathrm{Hom}(N(\mathbb{Y}), N(\mathbb{Z}))$. Now, $\mathrm{Hom}(\mathbb{Y}, \mathbb{Z}) \to \mathrm{Hom}(N(\mathbb{Y}), N(\mathbb{Z}))$ is on-to, thus the on-to property for $\mathrm{Hom}_{\mathcal{P}}(P, Q) \to \mathrm{Hom}_{\mathcal{P}'}(NP, NQ)$. We conclude that N is faithful. Similarly for M.

The natural transformations $\mathcal{P} \to \mathcal{Q}$ and $\mathcal{P}' \to \mathcal{Q}'$ are faithful, so we deduce that $\mathfrak{M}$ and $\mathfrak{N}$ are also faithful.

We prove now the fullness of the functors M and N. We show that there are natural transformations $MN \to \mathrm{id}_{\mathcal{P}}$ and $\mathrm{id}_{\mathcal{P}'} \to NM$. For this, we describe the compositions MN and NM.

We start with the analysis for MN. We fix an object P of $\mathcal{P}$, that is a pair $(\{\ Y_i \xrightarrow{f_i} Y\ \}, \{u_i\})$. We denote by u_i' the first projection from the fiber product $U' \times_U Y_i$ on U'. Then, the image of P via MN is the object

$$MN(P) := (\{\ U' \times_U Y_i \xrightarrow{f_i \circ p_i} Y\ \}, \{\alpha \circ u_i'\}).$$

In the category $\mathcal{P}$, we have an arrow from $MN(P)$ to P which is given by the p_i's. This provides a natural transformation $MN \to \mathrm{id}_{\mathcal{P}}$.

For the composition NM, let us take an object $P' := (\{\ Y_i' \xrightarrow{f_i'} Y\ \}, \{u_i'\})$. For the fiber product $U' \times_U Y_i'$ we denote by p_i' the the projection on Y_i' (which is etale by base-change) and by v_i' the projection on U'. Then, we have:

$$NM(P') := (\{\ U' \times_U Y_i' \xrightarrow{f_i' \circ p_i'} Y\ \}, \{v_i'\}).$$

In the sequel, we construct an arrow from P' to $NM(P')$. For this, we identify Y_i' with the fiber product $U' \times_{U'} Y_i'$ and the fiber product $U' \times_U Y_i'$ with $(U' \times_U U') \times_{U'} Y_i'$. Then, the diagonal map provides an arrow:

$$Y_i' \xrightarrow{\quad \Delta U' \times \mathrm{id}_{Y_i'} \quad} U' \times_U Y_i'\ .$$

Then, in the category $\mathcal{P}'$, we have an arrow given by the $\Delta U' \times \mathrm{id}_{Y_i'}$ between P' and $NM(P')$.

We conclude on the equivalence between $F_U^a(Y) := \mathcal{Q}$ and $F_{U'}^a(Y) := \mathcal{Q}'$ as follows. From the above properties for M and N, we deduce that $\mathfrak{M}$

and $\mathfrak{N}$ are fully faithful. So we have the isomorphisms $\mathfrak{M}\mathfrak{N} \simeq \mathrm{id}_Q$ and $\mathrm{id}_{Q'} \simeq \mathfrak{N}\mathfrak{M}$. This ends the proof. ∎

Remark 2.87. The previous result shows that there is a unique up to isomorphism stack associated to any pair (X, D). We denote it by $\mathcal{X}$ and call it the smooth Deligne-Mumford stack associated to the pair (X, D).

2.4.3 Reminders about sheaves on Deligne-Mumford stacks

This is a brief reminder regrading sheaves on Deligne-Mumford stacks, according to [27], [43], [26], [2].

Definition 2.88. *1. ([27], Definition 12.1, (ii) and Remark 12.1.2, (ii), "etale site on a Deligne-Mumford stack") Let $\mathcal{F}$ be a Deligne-Mumford stack. The etale site $Et(\mathcal{F})$ on $\mathcal{F}$ is defined by:*

(a) an object of $Et(\mathcal{F})$ is a pair (U, u), with U an $S-$scheme and $u : U \to \mathcal{F}$ an etale [representable] $1-$morphism; we call such an object an atlas;

(b) an arrow between two pairs (U, u) and (V, v) is a pair (φ, α), with $\varphi : U \to V$ a $1-$morphism of $S-$schemes and α a $2-$morphism between u and $v \times \varphi$.

A covering in $Et(\mathcal{F})$ is a family $\{(\varphi_i, \alpha_i) : (U_i, u_i) \to (U, u)\}_i$ such that $\sqcup_i \varphi_i : \sqcup_i U_i \to U$ is onto and etale.

2. ([43], 7.18, "sheaf on a Deligne-Mumford stack") Let $\mathcal{F}$ be a Deligne-Mumford stack. A quasi-coherent sheaf $\mathcal{S}$ on the etale site of $\mathcal{F}$ is the following data:

(a) for any atlas (U, u), a quasi-coherent sheaf $\mathcal{S}_U$ on U;

(b) for any arrow $(\varphi, \alpha) : (U, u) \to (V, v)$, an isomorphism

$$\theta_{\varphi,\alpha} = \theta_\varphi : \mathcal{S}_U \longrightarrow \varphi^* \mathcal{S}_V \ ,$$

such that the following **cocycle condition** *is satisfied: for any three atlases (U, u), (V, v) and (W, w) and any commutative diagram as in Figure 2.20 we have:*

$$\theta_{\psi\circ\varphi} = \theta_\varphi \circ \varphi^* \theta_\psi : \mathcal{S}_U \longrightarrow (\psi \circ \varphi)^* \mathcal{S}_W = \varphi^*(\psi^* \mathcal{S}_W). \qquad (2.4.3)$$

We say that $\mathcal{S}$ is coherent (vector bundle) if all $\mathcal{S}_U$ are coherent (locally free).

Remark 2.89. In [27], there is a slightly different approach for the definition of a sheaf on a Deligne-Mumford stack (see 12.2.1, (ii)). For an arrow $(\varphi, \alpha) : (U, u) \to (V, v)$, the authors consider instead of the isomorphism

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[dr, "u"'] & V \arrow[r, "\psi"] \arrow[d, "v"] & W \arrow[dl, "w"] \\ & \mathcal{F} & \end{tikzcd}$$

Figure 2.20: Commutative diagram for cocycle condition for sheaves on Deligne-Mumford stacks.

θ_φ, the isomorphism $\gamma_{\varphi,\alpha} : \mathcal{S}_V \longrightarrow \varphi_* \mathcal{S}_U$ and a similar co-cycle condition. Using the adjointness of $\bullet^*$ and $\bullet_*$, they deduce an arrow $\gamma^{\#}_{\varphi,\alpha} : \varphi^* \mathcal{S}_V \to \mathcal{S}_U$ and call a sheaf $\mathcal{S}$ on $\mathcal{F}$ cartesian if $\gamma^{\#}_{\varphi,\alpha}$ is an isomorphism for all arrows φ, α. In the definition we consider that all sheaves are cartesian. ♣

Example 2.90. Let M be a smooth complex variety of dimension n and consider a finite subgroup $G \subset SL_n(\mathbb{C})$ acting freely on M. To the quotient $X := M/G$ we can associate a smooth Deligne-Mumford stack $\mathcal{X}$, as in Section 2.4. Here we consider the divisor $D = 0$ and we take the morphism $\pi : M \to X = M/G$. Then, following [27], 12.4.6, the category of [cartesian] sheaves on the etale site of $\mathcal{X}$ is the same as the category of $G-$equivariant sheaves on M. ♣

Chapter 3

Equivalences of derived categories

Introduction

We use here the notations of Chapter 1. All over the chapter, S is a fixed base-scheme and n is a fixed integer and we denote by μ_{2^n-1} the cyclic group of roots of unity of order (2^n-1), acting on $\mathbb{A}^n$ as previously stated. In the sequel, we prove the following theorem:

Theorem 3.1. *The derived category of $\mu_{2^n-1}-$equivariant coherent sheaves on the affine space $\mathbb{A}^n$ is equivalent to the derived category of coherent sheaves on the $\mu_{2^n-1}-$Hilbert scheme of $\mathbb{A}^n$.*

The idea of the proof is to "translate" the previous equivalence at stack level. The derived category of coherent sheaves on the Deligne-Mumford stack associated to the orbifold quotient $\mathbb{A}^n/\mu_{2^n-1}$ is the same as the derived category of $\mu_{2^n-1}-$equivariant coherent sheaves on $\mathbb{A}^n$.

We recall that a divisorial contraction is defined as follows.

Definition 3.2. *[[22], Definition 1.4] A morphism is a divisorial contraction if it is a projective birational morphism with exceptional prime divisor.*

The resolution of singularities $f : \mu_{2^n-1}\text{-Hilb}\mathbb{A}^n \to \mathbb{A}^n/\mu_{2^n-1}$ is not a divisorial contraction, so Kawamata's result [22] can not be applied. We eliminate this inconvenience by proving that the previous resolution can be decomposed into a chain of divisorial contractions, such that at each step one can apply Kawamata's theorem. We conclude by help of some technical results on derived equivalences.

The chapter is organized as follows. The first section contains the description of the intermediate partial resolutions from $\mathbb{A}^n/\mu_{2^n-1}$ to μ_{2^n-1}-Hilb$\mathbb{A}^n$ providing a chain of divisorial contractions that split the resolution map f. The second section contains the machinery that allows us in Section 3.3 to conclude on the proof of Theorem 3.1.

3.1 From crepant resolution to divisorial contractions

We start with the following **recalls**:

1. For a positive integer we denote by $\pmod{n}$ the remainder of the division by n. For an integer i, the notation $i \pmod{n}$ stands for the unique non-negative integer j between 0 and $n-1$ such that $i-j$ is a multiple of n.

2. According to Notation 1.39, Chapter 1, we denote by h_n the vector $\frac{1}{2^n-1}(1, 2, 2^2, \ldots, 2^{n-1})$ and by $2^k \star h_n$, $1 \leq k \leq n-1$, the vector having on the l^{th} position the fraction $\frac{1}{2^n-1} 2^{(l-1+k) \pmod{n}}$.

3. We denote by $\{e_i\}_{1 \leq i \leq n}$ the canonical basis of $\mathbb{Z}^n$ and by σ_0 the cone generated by $e_1, \ldots, e_n$ in a lattice containing $\mathbb{Z}^n$.

4. We denote by N the lattice $\mathbb{Z}^n + h_n\mathbb{Z}$.

We describe here the algorithm allowing us to subdivide the crepant resolution $f : \mu_{2^n-1}$-Hilb$\mathbb{A}^n \to \mathbb{A}^n/\mu_{2^n-1}$ into a chain of partial resolutions

$$X = \mu_{2^n-1}-\mathrm{Hilb}\mathbb{A}^n \longrightarrow \cdots \xrightarrow{f_1} X_0 \xrightarrow{f_0} Y = \mathbb{A}^n/\mu_{2^n-1}. \tag{3.1.1}$$

The orbifold $\mathbb{A}^n/\mu_{2^n-1}$ is a toric variety with lattice $N = \mathbb{Z}^n + h_n\mathbb{Z}$ and fan the cone $\sigma_0 = \langle e_1, \ldots, e_n \rangle$. The desingularisation consists in subdividing the cone σ_0 until we recover all the cones of μ_{2^n-1}-Hilb$\mathbb{A}^n$. We construct a number of n partial resolutions, denoted X_i, for i in $\{0, \ldots, n-1\}$. Each such partial resolution X_i is a toric variety obtained from the previous partial resolution X_{i-1} by subdividing its fan by help of a primitive vector v_{i-1}. The primitive vector v_{i-1} provides a divisor which is the exceptional locus, so the toric morphism $f_i : X_i \to X_{i-1}$ is a divisorial contraction.

Definition 3.3. *Let X be a toric variety, σ a cone in the fan defining X and $X(\sigma)$ its corresponding affine piece. If $X(\sigma)$ is singular, we call the cone σ a singular cone, otherwise we call it a smooth cone.*

In the sequel, we need the following descriptive relation between a given vector $2^k \star h_n$, $1 \leq k \leq n-1$, and the junior lattice vectors $h_n, e_1, \ldots, e_{n-k}$.

Lemma 3.4. *Let k be an index between 1 and $n-1$. Then, we have:*

$$2^k \star h_n = \frac{1}{2^{n-k}} h_n + \frac{1}{2^{n-k}} e_1 + \frac{1}{2^{n-k-1}} e_2 + \cdots + \frac{1}{2} e_{n-k}. \tag{3.1.2}$$

Proof:

Clear, because $2^k \star h_n$ is the vector

$$\frac{1}{2^n-1}(2^k, 2^{k+1}, \dots, 2^{n-1}, \underbrace{1}_{(n-k)^{\text{th}} \text{ position}}, 2, \dots, 2^{k-1}).$$

∎

3.1.1 The first steps: $k = 0, 1, 2$

As previously stated, consider the quotient $\mathbb{A}^n/\mu_{2^n-1}$ as a toric variety with cone σ_0 and lattice $N = \mathbb{Z}^n + h_n\mathbb{Z}$. In order to obtain the varieties X_i, $i \in \{1, \dots, n\}$, we keep the lattice $N = \mathbb{Z}^n + h_n\mathbb{Z}$ unchanged and we proceed to a subdivision of the fan as follows.

To construct the partial resolution X_0, we add the vector h_n in the cone σ_0 and we split it into subcones. The resulting fan is the fan of X_0. Then, we proceed to the construction of X_1. We search among the $n-$dimensional cones of the fan of X_0 for those that contain the vector $2\star h_n$. We split them by help of $2 \star h_n$ and the resulting fan is the fan of X_1. In general, at Step k, the idea is to make use of the vector $2^k \star h_n$ to subdivide the fan of the previous partial resolution X_{k-1}. Let us see the process for the case when k is among $0, 1, 2$.

Step 0: We add the vector h_n and subdivide the cone σ_0 into n cones, by replacing each e_i by h_n:

$$\langle e_1, \dots, e_{i-1}, h_n, e_{i+1}, \dots, e_n\rangle, i \in \{1, \dots, n\}.$$

Thus, for $i = 1$ we get the cone $\langle h_n, e_2, \dots, e_n\rangle$ and for $i = n$ the cone $\langle h_n, e_1, \dots e_{n-1}\rangle$.

We denote by $\mathcal{C}_0$ the set of all those cones. We define X_0 to be the variety of lattice N and fan formed with the cones of $\mathcal{C}_0$.

Each of the cones of $\mathcal{C}_0$ has dimension n (it is easy to check that the n vectors composing it are independent). By Lemma 1.48, Section 1.3, Chapter 1, the cone

$$\langle h_n, e_2, \dots, e_n\rangle$$

is the only smooth cone. There are a total of $n - 1$ singular cones. We denote by $\mathcal{S}_0$ the set of all singular cones

$$\mathcal{S}_0 = \{\langle e_1, \dots, e_{i-1}, h_n, e_{i+1}, \dots, e_n\rangle, i \in \{2, \dots, n\}\}$$

and by $\mathcal{R}_0$ the set formed with the smooth cone

$$\mathcal{R}_0 = \{\langle h_n, e_2, \dots, e_n\rangle\}.$$

We remark that the cones of the set $\mathcal{C}_0$ do not overlap.

So far, we have constructed a partial desingularisation $f_0 : X_0 \to \mathbb{A}^n/\mu_{2^n-1}$. The variety X_0 is toric: its fan contains one smooth cone and $n-1$ singular ones.

Step 1: By Lemma 3.4, the vector $2\star h_n$ is in the singular $n-$dimensional cone $\langle h_n, e_1, \ldots, e_{n-1}\rangle$. Because the cones of $\mathcal{C}_0$ do not overlap, the primitive vector $2\star h_n$ is in no other cone, either singular or smooth.

Now, we subdivide the singular cone $\langle h_n, e_1, \ldots, e_{n-1}\rangle$ into n subcones by replacing in turn one of its vectors by $2\star h_n$ and keeping the other vectors unchanged. Let us denote by $\mathcal{NC}_1$ the set of the cones obtained by splitting $\langle h_n, e_1, \ldots, e_{n-1}\rangle$. Again by help of Lemma 1.48, we can conclude that we obtain two smooth cones, namely the ones obtained by replacing the vectors h_n, respectively e_1, by $2\star h_n$, that is the cones:

$$\langle 2\star h_n, e_1, \ldots, e_{n-1}\rangle$$

and

$$\langle h_n, 2\star h_n, e_2, \ldots, e_{n-1}\rangle.$$

We denote by $\mathcal{NR}_1$ the set formed with those two new smooth cones.

All other cones are $n-$dimensional singular cones. As previously, to prove they are $n-$dimensional it is enough check that the vectors composing them are linearly independent. We denote by $\mathcal{NS}_1$ the set of all singular cones obtained from $\langle h_n, e_1, \ldots, e_{n-1}\rangle$ by help of $2\star h_n$, by replacing each e_i, $i \geq 2$, by $2\star h_n$. Here, for $i = 2$ we get the cone $\langle e_1, 2\star h_n, e_3, \ldots, e_{n-1}, h_n\rangle$ and for $i = n-1$ the cone $\langle e_1, \ldots, e_{n-2}, 2\star h_n, h_n\rangle$. We thus have:

$$\mathcal{NS}_1 = \{\langle e_1, \ldots, e_{i-1}, 2\star h_n, e_{i+1}, \ldots, e_{n-1}, h_n\rangle, i \in \{2, \ldots, n-1\}\}.$$

We denote by $\mathcal{R}_1$ the set $\mathcal{R}_0 \cup \mathcal{NR}_1$. This is the set of all smooth cones after the subdivision of Step 1. We put:

$$\mathcal{LS}_0 = \{\langle e_1, \ldots, e_{i-1}, h_n, e_{i+1}, \ldots, e_n\rangle, i \in \{2, \ldots, n-1\}\}$$

to be the set of singular cones of the previous step from which we eliminate the cone $\langle h_n, e_1, \ldots, e_{n-1}\rangle$ that we just subdivided. This is:

$$\mathcal{LS}_0 = \mathcal{S}_0 \setminus \{\langle h_n, e_1, \ldots, e_{n-1}\rangle\}.$$

We define $\mathcal{S}_1$ to be the set $\mathcal{LS}_0 \cup \mathcal{NS}_1$. It is the set of all singular cones after the subdivision by help of $2\star h_n$. We also put $\mathcal{C}_1$ the union of $\mathcal{R}_1$ and $\mathcal{S}_1$:

$$\mathcal{C}_1 = \mathcal{R}_1 \cup \mathcal{S}_1.$$

The fan formed with the cones of $\mathcal{C}_1$ contains three smooth cones and a total of $[(n-1)-1] + (n-2) = 2(n-2)$ singular cones. We define a toric variety X_1 of fan $\mathcal{C}_1$ and lattice N. So far, we have a partial desingularisation X_1, with a map $f_1 : X_1 \to X_0$.

We also remark that the cones of $\mathcal{C}_1$ do not overlap. If σ and σ' are two cones of $\mathcal{C}_1$, three possibilities can occur. To see that, we write:

$$\mathcal{C}_1 = \mathcal{NC}_1 \cup (\mathcal{C}_0 \setminus \{\langle h_n, e_1, \ldots, e_{n-1}\rangle\}).$$

• If both cones σ and σ' are in $\mathcal{C}_0 \setminus \{\langle h_n, e_1, \ldots, e_{n-1}\rangle\}$, then by the proof at the end of Step 0, they do not overlap.

• If one cone is in $\mathcal{NC}_1$ and the other is in $\mathcal{C}_0 \setminus \{\langle h_n, e_1, \ldots, e_{n-1}\rangle\}$, let us suppose for a choice that σ is in $\mathcal{NC}_1$ and σ' is in $\mathcal{C}_0 \setminus \{\langle h_n, e_1, \ldots, e_{n-1}\rangle\}$. The cone σ is entirely contained in $\langle h_n,\ e_1, \ldots,\ e_{n-1}\rangle$. Again by the proof at the end of Step 0, $\langle h_n,\ e_1, \ldots,\ e_{n-1}\rangle$ and σ' don't overlap, so neither do σ and σ'.

• We suppose now that σ and σ' are both in $\mathcal{NC}_1$.

Let us describe more detailed the cones of $\mathcal{NC}_1$. First, we remark that the cone $\langle 2 \star h_n,\ e_1, \ldots,\ e_{n-1}\rangle$ doesn't contain the vector h_n. We argue by contradiction and suppose that h_n is a linear combination with non-negative coefficients a, a_i, $1 \leq i \leq n-1$, of the vectors $2 \star h_n,\ e_1, \ldots,\ e_{n-1}$:

$$h_n = a \cdot 2 \star h_n + a_1 \cdot e_1 + \cdots + a_{n-1} \cdot e_{n-1} =$$
$$= a(\frac{1}{2^{n-1}} h_n + \frac{1}{2^{n-1}} e_1 + \cdots + \frac{1}{2} e_{n-1}) + a_1 \cdot e_1 + \cdots + a_{n-1} \cdot e_{n-1},$$

where the last equality comes from Lemma 3.4. We use the fact that the vectors $h_n, e_1, \ldots, e_{n-1}$ are linearly independent. We identify the coefficient of h_n on the right hand side and left hand side of the above equality and deduce that a is positive. On the other hand, identifying the coefficient of e_{n-1} gives $a_{n-1} = -\frac{a}{2}$, which is a contradiction.

Secondly, let us consider a cone of the form $\langle h_n,\ e_1, \ldots,\ e_{i-1},\ 2 \star h_n,\ e_{i+1}, \ldots,\ e_{n-1}\rangle$, for some index $1 \leq i \leq n-1$. (Here, for $i = 1$ we have the cone $\langle h_n, 2 \star h_n,\ e_2, \ldots, e_{n-1}\rangle$ and for $i = n-1$ the cone $\langle h_n,\ e_1, \ldots,\ e_{n-2},\ 2 \star h_n\rangle$). Then, the vector e_i is not in such a cone. To see this, consider a liner combination with non-negative coefficients

$$e_i = a \cdot h_n + b \cdot 2 \star h_n + \sum_{j \neq i,n} a_j \cdot e_j.$$

Identifying the n^{th} coordinate on the right hand side and left hand side provides $0 = a \cdot \frac{2^{n-1}}{2^n - 1} + b \cdot \frac{1}{2^n - 1}$, which gives $a = b = 0$. This leads to a contradiction (the vector e_i can not be a linear combination of the vectors e_j, $j \neq i$).

Now, we can conclude as follows. If σ is the cone $\langle 2 \star h_n,\ e_1, \ldots,\ e_{n-1}\rangle$ and σ' any of the cones $\langle h_n,\ e_1, \ldots,\ e_{i-1},\ 2 \star h_n,\ e_{i+1}, \ldots,\ e_{n-1}\rangle$, for some index i, then h_n is in σ', but not in σ and e_i is in σ, but not in σ'. If we have $\sigma = \langle h_n,\ e_1, \ldots,\ e_{i-1},\ 2 \star h_n,\ e_{i+1}, \ldots,\ e_{n-1}\rangle$, for some index i, and $\sigma' = \langle h_n,\ e_1, \ldots,\ e_{i'-1},\ 2 \star h_n,\ e_{i'+1}, \ldots,\ e_{n-1}\rangle$, for some index $i' > i$, then

e_i is in σ', but not in σ, while $e_{i'}$ is in σ, but not in σ'. In both cases, we conclude that neither σ and σ' do not overlap.

Step 2: We consider now the vector $2^2 \star h_n$. By Lemma 3.4, $2^2 \star h_n$ is a convex combination of the vectors $h_n, e_1, \ldots, e_{n-2}$ (see also Figure 3.1). Two of the singular cones of the previous step contain these vectors: $\sigma_1 := \langle e_1, \ldots, e_{n-2}, h_n, e_n \rangle$ and $\sigma_2 := \langle e_1, \ldots, e_{n-2}, h_n, 2 \star h_n \rangle$. As before, because the cones of $\mathcal{C}_1$ do not overlap, we conclude that the previous two cones are the only cones that contain $2^2 \star h_n$.

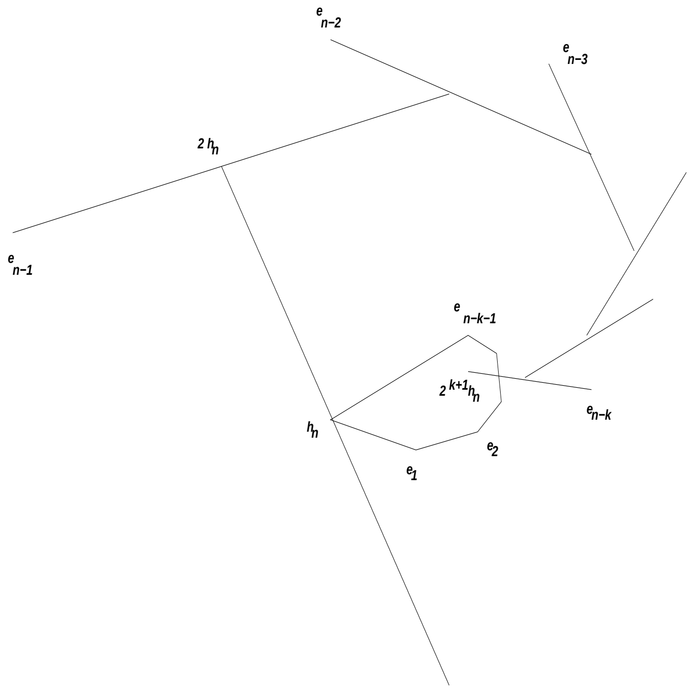

Figure 3.1: Vector $2^k \star h_n$ as barycenter of face $\langle e_1, \ldots, e_{n-k}, h_n \rangle$.

We can subdivide the face $\langle e_1, \ldots, e_{n-2}, h_n \rangle$ into $n-1$ sub-cones, each of dimension $n-1$, obtained by replacing one of the vectors $e_1, \ldots, e_{n-2}$ or h_n by $2^2 \star h_n$. Let us denote them by $\tau_1, \ldots, \tau_{n-1}$. This gives us the idea how to subdivide σ_1, respectively σ_2.

First, let us consider the cone $\sigma_1 = \langle e_1, \ldots, e_{n-2}, h_n, e_n \rangle$. It is easy to see that the vector e_n is outside the face $\langle e_1, \ldots, e_{n-2}, h_n \rangle$. Then, we take each of the cones τ_i as a basis and e_n as a summit of a $n-$dimensional cone. We construct in this way a total of $n-1$ cones, from which two are smooth and the rest are singular. By Lemma 1.48, the smooth cones are the one obtained by replacing h_n, respectively e_1, by $2^2 \star h_n$, this is

$$\langle e_1, \ldots, e_{n-2}, 2^2 \star h_n, e_n \rangle,$$

respectively

$$\langle 2^2 \star h_n, e_2, \ldots, e_{n-2}, h_n, e_n \rangle.$$

It is easy to prove that all singular cones are $n-$dimensional (take a null linear combination of the vectors of a cone, use the description of $2^2 \star h_n$ by Lemma 3.4 and get that all coefficients are zero). Such a singular cone is of the form

$$\langle e_1, \ldots, e_{i-1}, 2^2 \star h_n, e_{i+1}, \ldots, e_{n-2}, h_n, e_n \rangle,$$

for i in $\{2, \ldots, n-2\}$. We remark that e_1 and h_n are always in such a cone. Here, for $i = n-2$ we get the cone $\langle e_1, \ldots, e_{n-3}, 2^2 \star h_n, h_n, e_n \rangle$. We remark also that if we replace in the cone $\langle e_1, \ldots, e_{n-2}, h_n, e_n \rangle$ the vector e_n by $2^2 \star h_n$, we obtain no $n-$dimensional cone. Indeed, that comes from Lemma 3.4, because there is one linear relation between the vectors $e_1, \ldots, e_{n-2}, h_n, 2^2 \star h_n$.

We know that the cones σ_1 and σ_2 do not overlap (they are both cones of $\mathcal{C}_1$). In particular, none of the cones $\langle e_1, \ldots, e_{i-1}, 2^2 \star h_n, e_{i+1}, \ldots, e_{n-2}, h_n, e_n \rangle$, for some index $2 \leq i \leq n-2$, and σ_2 do overlap. That allows us to subdivide the cone $\sigma_2 = \langle e_1, \ldots, e_{n-2}, h_n, 2 \star h_n \rangle$ by help of vector $2^2 \star h_n$, without fear of recovering some previously created cones.

The geometric reason for this is actually that $2 \star h_n$ and e_n exclude reciprocally each other: relation $\frac{1}{2}(2 \star h_n) + \frac{1}{2} e_n = h_n$ says that $2 \star h_n$ and e_n are in opposite hyperplanes with respect to h_n. We deduce that $2 \star h_n$ and e_n are in the two opposite parts of the “world according to” $e_1, \ldots, e_{n-2}, h_n$.

Thus, it makes sense to subdivide the cone $\langle e_1, \ldots, e_{n-2}, h_n, 2 \star h_n \rangle$ using the vector $2^2 \star h_n$. There occurs two more smooth cones, obtained as before by replacing h_n (respectively e_1) and $n-3$ singular cones of dimension n. The smooth cones obtained from σ_2 are:

$$\langle e_1, \ldots, e_{n-2}, 2 \star h_n, 2^2 \star h_n \rangle,$$

respectively

$$\langle 2^2 \star h_n, e_2, \ldots, e_{n-2}, h_n, 2 \star h_n \rangle.$$

The singular cones obtained from σ_2 are:

$$\langle e_1, \ldots, e_{i-1}, 2^2 \star h_n, e_{i+1}, \ldots, e_{n-2}, h_n, 2 \star h_n \rangle,$$

for i in $\{2, \ldots, n-2\}$. Here, for $i = n-2$ we obtain the cone $\langle e_{1,d} ots, e_{n-3}, 2^2 \star h_n, h_n, 2 \star h_n \rangle$.

Let us give a list of the cones that occurred up to now, at the end of Step 2. We have a total of $1 + 2 + 2^2 = 2^3 - 1$ smooth cones that are:

• the three smooth cones of $\mathcal{R}_1$

• the four new smooth cones obtained while subdividing by help of $2^2 \star h_n$, that is the cones of the following set:

$$\begin{aligned}\mathcal{NR}_2 = \{&\langle e_1, \ldots, e_{n-2}, 2^2 \star h_n, e_n\rangle \\ &\langle 2^2 \star h_n, e_2, \ldots, e_{n-2}, h_n, e_n\rangle \\ &\langle e_1, \ldots, e_{n-2}, 2 \star h_n, 2^2 \star h_n\rangle \\ &\langle 2^2 \star h_n, e_2, \ldots, e_{n-2}, h_n, 2 \star h_n\rangle\}\end{aligned}$$

We denote by $\mathcal{R}_2$ the set of all smooth cones at the end of Step 3, that is the set $\mathcal{R}_1 \cup \mathcal{NR}_2$.

Let us denote by $\mathcal{NS}_2$ the set of all singular cones constructed by subdivision by help of $2^2 \star h_n$, that is the set:

$$\begin{aligned}&\mathcal{NS}_2 = \mathcal{NS}_{2,1} \cup \mathcal{NS}_{2,2} := \\ &= \{\langle e_1, \ldots, e_{i-1}, 2^2 \star h_n, e_{i+1}, \ldots, e_{n-2}, h_n, e_n\rangle, i \in \{2, \ldots, n-2\}\} \\ &\cup \{\langle e_1, \ldots, e_{i-1}, 2^2 \star h_n, e_{i+1}, \ldots, e_{n-2}, h_n, 2 \star h_n\rangle, i \in \{2, \ldots, n-2\}\}\end{aligned}$$

We also denote by $\mathcal{LS}_2$ the set $\mathcal{S}_1 \setminus \{\sigma_1, \sigma_2\}$. We put $\mathcal{S}_2 = \mathcal{LS}_2 \cup \mathcal{NS}_2$. We have a total of $[2(n-2) - 2] + 2(n-3) = 2^2(n-3)$ singular cones.

We put $\mathcal{C}_2 = \mathcal{R}_2 \cup \mathcal{S}_2$. One can prove that the cones of the set $\mathcal{C}_2$ do not overlap (see the next section for a proof). We define the variety X_2 to be the toric variety of lattice $N = \mathbb{Z}^n + h_n\mathbb{Z}$ and fan formed with the cones of $\mathcal{C}_2$. We get a partial resolution $f_2 : X_2 \to X_1$, which is a divisorial contraction. The exceptional locus is provided by the divisor given by the junior lattice vector $2^2 \star h_n$.

To resume, the whole construction of the set $\mathcal{C}_2$ of Step 2 can be described as follows. By Lemma 3.4, $2^2 \star h_n$ is a barycenter for $\{e_1, \ldots, e_{n-2}, h_n\}$. We subdivide the face $\langle e_1, \ldots, e_{n-2}, h_n\rangle$ into $n-1$ cones τ_i, by replacing e_i, $1 \leq i \leq n-2$ (respectively h_n, for τ_{n-1}) by $2^2 \star h_n$. Now, the vectors $2 \star h_n$ and e_n are on opposite sides of the face $\langle e_1, \ldots, e_{n-2}, h_n\rangle$. We consider all the cones $\langle \tau_i, 2 \star h_\rangle$, τ_i, e_n, $1 \leq i \leq n-1$. This gives a collection of cones which is nothing else but $\mathcal{NR}_2 \cup \mathcal{NS}_2$.

In the sequel, we fix an integer k and describe Step k of the algorithm.

3.1.2 Summary of construction at Step s, for $s \leq k$ and consistency of the algorithm

At Step $s \leq k$, we constructed a toric variety X_s as follows. Let X_{s-1} denote the toric variety constructed at Step $s-1$. The lattice of X_{s-1} is $N = \mathbb{Z}^n + h_n\mathbb{Z}$ and its fan is formed with the cones of a set denoted $\mathcal{C}_{s-1}$. The set $\mathcal{C}_{s-1}$ contains only $n-$dimensional cones. We denote by $\mathcal{S}_{s-1}$ the set of all singular cones and by $\mathcal{R}_{s-1}$ the set of all smooth cones of $\mathcal{C}_{s-1}$. We keep unchanged the lattice N and subdivide the fan of X_{s-1} by help of the vector $2^s \star h_n$.

For that, let us denote by $\mathcal{A}_s$ the set of all $n-$dimensional cones of $\mathcal{C}_{s-1}$ that contain $2^s \star h_n$. The set $\mathcal{A}_s$ is a subset of $\mathcal{S}_{s-1}$. We denote by $\mathcal{LS}_s$ the set of lasting singular cones, after taking the set $\mathcal{A}_s$, that is the set:

$$\mathcal{LS}_s = \mathcal{S}_{s-1} \setminus A_s.$$

We subdivide each cone of the set $\mathcal{A}_s$ by help of $2^s \star h_n$. We denote by $\mathcal{NC}_s$ the set of the $n-$dimensional cones that we obtain by the subdivision process. The set $\mathcal{NC}_s$ is partitioned into the set of singular cones denoted $\mathcal{NS}_s$ and the set of smooth cones denoted $\mathcal{NR}_s$:

$$\mathcal{NC}_s = \mathcal{NS}_s \sqcup \mathcal{NR}_s.$$

We put $\mathcal{LC}_s$ to be the set $\mathcal{C}_{s-1}$ from which we removed the set $\mathcal{A}_s$, that is:

$$\mathcal{LC}_s = \mathcal{C}_{s-1} \setminus \mathcal{A}_s = (\mathcal{S}_{s-1} \setminus \mathcal{A}_s) \sqcup \mathcal{R}_{s-1}.$$

We also introduce the following notations:

• the set of all singular cones obtained at the end of Step s is denoted by $\mathcal{S}_s$, that is:

$$\mathcal{S}_s = \mathcal{LS}_s \cup \mathcal{NS}_s.$$

• the set of all smooth cones obtained at the end of Step s is denoted by $\mathcal{R}_s$, that is:

$$\mathcal{R}_s = \mathcal{R}_{s-1} \cup \mathcal{NR}_s.$$

• the set of all cones obtained at the end of Step s is denoted by $\mathcal{C}_s$, that is:

$$\mathcal{C}_s = \mathcal{R}_s \cup \mathcal{S}_s.$$

We remark that we have

$$\mathcal{C}_s = \mathcal{NC}_s \cup (\mathcal{C}_{s-1} \setminus \mathcal{A}_s).$$

We define the toric variety X_s to be the variety of lattice N and fan formed with the cones of $\mathcal{C}_s$. We have a toric map $f_s : X_s \to X_{s-1}$, with exceptional locus formed with the divisor provided by the vector $2^s \star h_n$.

Let $\#\mathcal{A}_s = t$, for some positive integer t and denote by $\sigma_{s,i}$, $1 \leq i \leq t$, the cones of $\mathcal{A}_s$. The subdivision process consists in replacing each $\sigma_{s,i}$ by a number of smooth and singular $n-$dimensional cones. We denote by $\mathcal{NS}_{s,i}$ (respectively $\mathcal{NR}_{s,i}$) is the set of the new singular (respectively smooth) cones created by the subdivision of $\sigma_{s,i}$. We have:

$$\mathcal{NS}_s = \bigcup_{i=1}^{t} \mathcal{NS}_{s,i}$$

and

$$\mathcal{NR}_s = \bigcup_{i=1}^{t} \mathcal{NR}_{s,i}.$$

In what follows, we detail this algorithm at Step k and prove:

Theorem 3.5. *1. The cones of the set $\mathcal{C}_k$ are $n-$dimensional cones given by the following:*

List $\mathcal{C}_k$

(a) smooth cones, forming the set $\mathcal{R}_k$:

$$\langle h_n, e_2, \ldots, e_{n-k}, w_{n-k+1}, \ldots, w_n \rangle$$

or

$$\langle e_1, e_2, \ldots, e_{n-k}, w_{n-k+1}, \ldots, w_n \rangle,$$

where, for any index j, $n-k+1 \leq j \leq n$, the vector w_j is either e_j or $2^{n-j+1} \star h_n$ and we don't consider the cone $\sigma_0 = \langle e_1, \ldots, e_n \rangle$.

(b) singular cones, forming the set $\mathcal{S}_k$:

$$\langle e_1, e_2, \ldots, e_{i-1}, h_n, e_{i+1}, \ldots, e_{n-k}, w_{n-k+1}, \ldots, w_n \rangle,$$

for an integer i, such that $2 \leq i \leq n-k$ where, for any index j, $n-k+1 \leq j \leq n$, the vector w_j is either e_j or $2^{n-j+1} \star h_n$.

The cones of the List $\mathcal{C}_k$ do not overlap, in the sense that two cones of List $\mathcal{C}_k$ have in common at most a $(n-1)-$dimensional face.

2. The cones of the set $\mathcal{A}_k$ are $n-$dimensional cones given by the following:

List $\mathcal{A}_k$

$$\langle e_1, \ldots, e_{n-k}, h_n, w_{n-k+2}, \ldots, w_n \rangle,$$

where, for any index j, $n-k+1 \leq j \leq n$, the vector w_j is either e_j or $2^{n-j+1} \star h_n$.

The proof of this theorem goes by recurrence. The cases $k = 0$ and $k = 1$, and the description of List $\mathcal{C}_2$ without the proof of no overlapping are provided in Section 3.1.1. We have to prove that if the above theorem is true for any non-negative integer $s < k$, then we recover the result for k also. Before giving the proof, we state the following results providing the consistency of the algorithm.

Corollary 3.6. *We have* $\#\mathcal{R}_k = 2^{k+1} - 1$.

Proof:

We use 1a of the List $\mathcal{C}_k$ in Theorem 3.5. In a smooth cone, the first $n-k$ positions are fixed, either $h_n, e_2, \ldots, e_{n-k}$ or $e_1, e_2, \ldots, e_{n-k}$. For any of the vectors w_j, for $n-k+1 \le j \le n$ we have the choice between e_j or $2^{n-j+1} \star h_n$. Thus, a total of $2 \cdot 2^{n-(n-k+1)+1} - 1 = 2^{k+1} - 1$. Here, we don't consider the cone σ_0, thus the -1 term above. ■

Corollary 3.7. *We have* $\#\mathcal{S}_k = 2^k(n-k-1)$.

Proof:

We use 1b of the List $\mathcal{C}_k$ in Theorem 3.5. For a fixed index i, we have $2^{n-(n-k+1)+1} = 2^k$ possible choices for $\{w_j,\ n-k+1 \le j \le n\}$. Now, the index i varies in the set $\{2, \ldots, n-k\}$. We conclude that the number of singular cones is as wanted $\#\mathcal{S}_k = 2^k(n-k-1)$. ■

Corollary 3.8. *(consistency of the algorithm) At the last step of the algorithm (i.e. for* $k = n-1$*) we recover* μ_{2^n-1}*-Hilb*$\mathbb{A}^n$.

Proof:

At Step $n-1$, we have a total of $2^n - 1$ smooth cones and no singular cone because $2^{n-1}(n-(n-1)-1) = 0$. We apply Corollary 1.36, Section 1.2.2, Chapter 1 to conclude. ■

The rest of the section is dedicated to the proof of Theorem 3.5 and is organized as follows. First, we prove that any cone in the List $\mathcal{C}_s$ is $n-$dimensional, for any $s \le k$. In particular, we deduce that any cone of the form $\langle e_1, \ldots, e_{i-1}, h_n, e_{i+1}, w_{i+2}, \ldots, w_n \rangle$, for w_j either e_j or $2^{n-j+1} \star h_n$ is also $n-$dimensional. Then, we prove that under the induction hypothesis – that is if the List $\mathcal{C}_{k-1}$ is of the form 1, Theorem 3.5, with no overlapping – we obtain the cones of the List $\mathcal{A}_k$. We then describe how to subdivide such a cone. We obtain the List $\mathcal{C}_k$ and in particular we deduce that the cones of the List $\mathcal{C}_k$ do not overlap. This ends the proof.

3.1.2.1 Induction hypothesis on $\mathcal{C}_s$, for $s < k$

In the sequel, in all following subsections, we suppose that the following hypothesis of recurrence is true. At Step s, for any $s < k$, the List $\mathcal{C}_s$ is given as in 1, of Theorem 3.5, that is:

List $\mathcal{C}_s$

1. smooth cones, forming the set $\mathcal{R}_s$:

$$\langle h_n, e_2, \ldots, e_{n-s}, w_{n-s+1}, \ldots, w_n \rangle$$

or

$$\langle e_1, e_2, \ldots, e_{n-s}, w_{n-s+1}, \ldots, w_n \rangle,$$

where, for any index j, $n-s+1 \leq j \leq n$, the vector w_j is either e_j or $2^{n-j+1} \star h_n$;

2. singular cones, forming the set $\mathcal{S}_s$:

$$\langle e_1, e_2, \ldots, e_{i-1}, h_n, e_{i+1}, \ldots, e_{n-s}, w_{n-s+1}, \ldots, w_n \rangle,$$

for an integer i, such that $2 \leq i \leq n-s$ where, for any index j, $n-s+1 \leq j \leq n$, the vector w_j is either e_j or $2^{n-j+1} \star h_n$.

Remark 3.9. Another description of List $\mathcal{C}_s$ is the following:

1. $\langle e_1, \ldots, e_{i-1}, \underbrace{h_n}_{\text{position } i}, e_{i+1}, \ldots, e_n \rangle$, for $i \in \{1, \ldots, n-s\}$.

   For $i = 1$ the corresponding cone is smooth, while for $i \geq 2$ the corresponding cone is singular.

2. $\langle e_1, \ldots, e_{i-1}, \underbrace{2 \star h_n}_{\text{position } i}, e_{i+1}, \ldots, e_{n-1}, \underbrace{h_n}_{\text{position } n} \rangle$, for index $i \in \{1, \ldots, n-s\} \cup \{n\}$.

   For $i = 1$ the corresponding cone is $\langle 2 \star h_n, e_2 \ldots, e_{n-1}, h_n \rangle$.

   For $i = n$, the corresponding cone is $\langle e_1, e_2 \ldots, e_{n-1}, 2 \star h_n \rangle$.

   They are both smooth. Any other cone is singular.

3. $\langle e_1, \ldots, e_{i-1}, \underbrace{2^2 \star h_n}_{\text{position } i}, e_{i+1}, \ldots, e_{n-2}, \underbrace{h_n}_{\text{position } n-1}, w_n \rangle$, for $i \in \{1, \ldots, n-s\} \cup \{n-1\}$ and $w_n \in \{e_n, 2 \star h_n\}$.

   For $i = 1$, respectively $i = n-1$, we have the smooth cones:

$$\langle 2^2 \star h_n, e_2, \ldots, e_{n-2}, h_n, e_n \rangle$$

   and

$$\langle e_1, e_2, \ldots, e_{n-2}, 2^2 \star h_n, e_n \rangle$$

4. For any integer $t \leq s$ we consider the cones

$$\langle e_1, \ldots, e_{i-1}, \underbrace{2^t \star h_n}_{\text{position } i}, e_{i+1}, \ldots, e_{n-t}, \underbrace{h_n}_{\text{position } n-t+1}, w_{n-t+2}, \ldots, w_n \rangle,$$

   where $i \in \{1, \ldots, n-s\} \cup \{n-t+1\}$ and for any $n-t+2 \leq j \leq n$ the vector w_j is in $\{e_j, 2^{n-j+1} \star h_n\}$.

   For $i = 1$ and $i = n-t+1$, the corresponding cones are smooth:

$$\langle 2^t \star h_n, e_2, \ldots, e_{n-t}, h_n, w_{n-t+2}, \ldots, w_n \rangle;$$

and

$$\langle e_1, \ldots, e_{n-t}, 2^t \star h_n, w_{n-t+2}, \ldots, w_n \rangle.$$

Any other cone is singular. ♣

For the first steps of the recurrence, the form of the cones in List $\mathcal{C}_s$ is accurate as proved in Section 3.1.1. We characterize in the sequel the cones of the set $\mathcal{C}_s$, for $s < k$.

Lemma 3.10. *The cones of List $\mathcal{C}_s$ are $n-$dimensional.*

Proof:

We prove that the n vectors composing such a cone are linearly independent. Thus, let us take a cone of the form $\langle e_1, \ldots, e_{i-1}, \underbrace{h_n}_{\text{position } i}, e_{i+1}, \ldots, e_{n-t}, \underbrace{2^t \star h_n}_{\text{position } n-t+1}, w_{n-t+2}, \ldots, w_n \rangle$, for some indices $1 \leq t \leq s$, and $1 \leq i \leq n-s$, or $i = n-t$, and such that for any j with $n-t+2 \leq j \leq n$, the vector w_j is either e_j or $2^{n-j+1} \star h_n$. Suppose we have a linear combination:

$$\begin{aligned} a_1 e_1 + \cdots + a_{i-1} e_{i-1} + a_i h_n + a_{i+1} e_{i+1} + \ldots \\ \cdots + a_{n-t} e_{n-t} + a_{n-t+1} 2^t \star h_n + a_{n-t+2} w_{n-t+2} + \cdots + a_n w_n = 0 \end{aligned}$$

Identifying coordinate by coordinate the two members of the above equality, we obtain a linear $n \times n$ system with unknowns $a_l, 1 \leq l \leq n$. Let us denote by α the positive integer $2^n - 1$ and for any index $l \geq n-t+2$, by δ_l the Kronecker symbol of index (w_l, e_l), this if $w_l = e_l$ and zero otherwise, i.e. if $w_l = 2^{n-l+1} \star h_n$.

Then, after some calculation, we obtain the following:

1. coordinate i is:

$$a_i + 2^t a_{n-t+1} + (1 - \delta_{n-t+2}) 2^{t-1} a_{n-t+2} + \cdots + (1 - \delta_n) 2 a_n = 0 \quad (3.1.3)$$

2. coordinate $n-t+1$ is:

$$a_i 2^{-n+t} + a_{n-t+1} + (1 - \delta_{n-t+2}) 2^{t-1} a_{n-t+2} + \cdots + (1 - \delta_n) 2 a_n = 0 \quad (3.1.4)$$

3. for an integer $l \leq n-t, l \neq i$, the corresponding coordinate is:

$$\alpha a_l 2^{-l+1} + a_i + a_{n-t+1} + (1 - \delta_{n-t+2}) 2^{t-1} a_{n-t+2} + \cdots + (1 - \delta_n) 2 a_n = 0 \quad (3.1.5)$$

This gives $a_{n-t+1} = 0$ and also $a_l = 0$, for any $l \leq n-t$. This means that the initial null combination reduces to a null combination between vectors h_n and $w_{n-t+2}, \ldots, w_n$. The discriminant of this system is:

$$\begin{vmatrix} 1 & (1-\delta_{n-t+2})2^{t-1} & (1-\delta_{n-t+3})2^{t-2} & \dots & (1-\delta_{n-1})2^2 & (1-\delta_n)2 \\ 2^{n-t+1} & \alpha\delta_{n-t+2}+(1-\delta_{n-t+2}) & (1-\delta_{n-t+3})2^{n-1} & \dots & (1-\delta_{n-1})2^{n-t+3} & (1-\delta_n)2^{n-t+2} \\ 2^{n-t+2} & (1-\delta_{n-t+2})2 & \alpha\delta_{n-t+3}+(1-\delta_{n-t+3}) & \dots & (1-\delta_{n-1})2^{n-t+4} & (1-\delta_n)2^{n-t+3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2^{n-2} & (1-\delta_{n-t+2})2^{t-3} & (1-\delta_{n-t+3})2^{t-4} & \dots & \alpha\delta_{n-1}+(1-\delta_{n-1}) & (1-\delta_n)2^{n-1} \\ 2^{n-1} & (1-\delta_{n-t+2})2^{t-2} & (1-\delta_{n-t+3})2^{t-3} & \dots & (1-\delta_{n-1})2 & \alpha\delta_n+(1-\delta_n) \end{vmatrix}. \tag{3.1.6}$$

Making zero on the first column and developing afterward gives that the determinant is the product $\alpha^{t-1}\Pi_{j=n-t+2}^{n}(2\delta_j-1)$. Independent on the choice for $w_j, j \geq n-k+1$ this is non zero. Thus, the system has a unique zero solution, which ends the proof. ■

Remark 3.11. In the above lemma, we didn't use anywhere the fact that we are at Step s. Thus the following corollary. ♣

Corollary 3.12. *For any index $i \geq 1$, and for any choice of $w_j \in \{e_j, 2^{n-j+1}\star h_n\}$, $i+1 \leq j \leq n$, the cone $\langle e_1, \dots, e_{i-1}, h_n, w_{i+1}, \dots, w_n\rangle$, is $n-$dimensional.* ■

3.1.2.2 Cones containing $2^k \star h_n$, the List $\mathcal{A}_k$

In the sequel, we suppose that the following induction hypothesis is true: for any $s < k$, the description of List $\mathcal{A}_s$, provided by 2, Theorem 3.5 holds.

Remark 3.13. The list $\mathcal{A}_s$, $s < k$, can also be described as follows.

List $\mathcal{A}_s$ — another form

1. one singular cone from the ones that first occurred at Step 0, this is the cone $\langle e_1, \dots, e_{n-s}, h_n, e_{n-s+2}, \dots, e_n\rangle$.

2. one singular cone from the cones that first occurred at Step 1, this is the cone $\langle e_1, \dots, e_{n-s}, h_n, e_{n-s+2}, \dots, e_{n-1}, 2\star h_n\rangle$.

3. two singular $n-$dimensional cones that first occurred at Step 2 :
$$\langle e_1, \dots, e_{n-s}, h_n, e_{n-s+2}, \dots, e_{n-2}, 2^2\star h_n, e_n\rangle,$$
and
$$\langle e_1, \dots, e_{n-s}, h_n, e_{n-s+2}, \dots, e_{n-2}, 2^2\star h_n, 2\star h_n\rangle.$$

4. In general, for $t < s$, there are 2^{t-1} singular $n-$dimensional cones containing $2^s\star h_n$ and that first occurred at Step t. We denote such a cone by $\sigma_{t,r}$, with t denoting the step when the cone first occurred and $1 \leq r \leq 2^{t-1}$ the range among the 2^{t-1} cones of the same step (with the convention that if $s=0$, then $r=1$). Such a cone is of the form
$$\langle e_1, \dots, e_{n-s}, h_n, e_{n-s+2}, \dots, e_{n-t}, 2^t\star h_n, w_{n-t+2}, \dots, w_n\rangle,$$
where on position $j \geq n-t+2$, the vector w_j is either e_j or $2^{n-j+1}\star h_n$. ♣

For an integer $s < k$, the vector $2^s \star h_n$ is in exactly 2^{s-1} singular $n-$dimensional cones. We prove now that under the induction hypothesis (i.e. description of $\mathcal{C}_s$ and $\mathcal{A}_s$, for $s < k$, as in Theorem 3.5), we recover the stated form for $\mathcal{A}_k$.

Lemma 3.14. *A cone σ is in the List $\mathcal{A}_k$ if and only if it contains the vectors h_n, $e_1, \dots, e_{n-k}$.*

Proof:

If a cone σ contains the vectors h_n, $e_1, \dots, e_{n-k}$, we apply Lemma 3.4 and we conclude that the vector $2^k \star h_n$ is in σ. Thus σ is in $\mathcal{A}_s$.

Let us suppose now that σ is in $\mathcal{A}_k$, that is σ is a cone of $\mathcal{C}_{k-1}$ containing the vector $2^k \star h_n$. We write $\mathcal{C}_{k-1}$ as a union of smooth and singular cones $\mathcal{C}_{k-1} = \mathcal{R}_{k-1} \cup \mathcal{S}_{k-1}$.

First we prove that σ can not be a smooth cone, i.e. σ is not in $\mathcal{R}_{k-1}$. By Lemma 1.48, Section 1.3, Chapter 1 we know that a smooth cone has on position $n-k+1$ either the vector e_{n-k+1} or $2^k \star h_n$. But, according to the induction hypothesis for the description of the cones of $\mathcal{C}_{k-1}$, all the cones of $\mathcal{R}_{k-1}$ have e_{n-k+1} on position $n-k+1$, so they can not contain $2^k \star h_n$. Thus, the set $\mathcal{A}_k$ is contained in the set of singular cones $\mathcal{S}_{k-1}$.

Now, using the alternate description of the List $\mathcal{C}_{k-1}$, such a cone is of the form $\langle e_1, \dots, e_{i-1}, \underbrace{2^t \star h_n}_{\text{position } i}, e_{i+1}, \dots, e_{n-t}, \underbrace{h_n}_{\text{position } n-t+1}, w_{n-t+2}, \dots, w_n \rangle$, for some $i \le n-k-1$ and vectors $w_j \in \{e_j, 2^{n-j+1} \star h_n\}$, $n-t+2 \le j \le n$. This means that we have a linear combination with non-negative coefficients a_i :

$$\begin{aligned} 2^k \star h_n &= a_1 e_1 + \dots + a_{i-1} e_{i-1} + a_i (2^t \star h_n) + a_{i+1} e_{i+1} + \dots \\ &\dots + a_{n-t} e_{n-t} + a_{n-t+1} h_n + a_{n-t+2} w_{n-t+2} + \dots + a_n w_n. \end{aligned}$$

Let us denote by α the integer $2^n - 1$ and by δ_j, for $j \ge n-k+1$, the Kronecker symbol of index (w_j, e_j). We identify coordinate by coordinate the two vectors above and we get a linear system with unknowns a_l, such that:

1. coordinate l, for $1 \le l \le n-t, l \ne i$ is:

$$\begin{aligned} \alpha a_l + 2^{l+t-1} a_i + 2^{l-1} a_{n-t+1} + \\ +(1-\delta_{n-t+2}) 2^{l+t-2} a_{n-t+2} + \dots \\ \dots + (1-\delta_n) 2^l a_n = 2^{(l+k-1) \pmod n}, \end{aligned} \tag{3.1.7}$$

   where $(l+k-1) \pmod n$ is $l+k-1$ for $l \le n-k$ and $l+k-1-n$ otherwise;

2. coordinate i is

$$\begin{aligned} 2^{i+t-1}a_i + 2^{i-1}a_{n-t+1} + \\ +(1-\delta_{n-t+2})2^{i+t-2}a_{n-t+2} + \dots \\ \dots + (1-\delta_n)2^i a_n = 2^{i+k-1}. \end{aligned} \tag{3.1.8}$$

3. coordinate l, for $l \geq n-t+1$ is:

$$\begin{aligned} 2^{l+t-1-n}a_i + 2^{l-1}a_{n-t+1} + \\ +(\delta_{l,n-t+2}\delta_{n-t+2}\alpha + (1-\delta_{n-t+2})2^{(l+t-2) \pmod n})a_{n-t+2} + \dots \\ \dots + (\delta_{l,n}\delta_n\alpha + (1-\delta_n)2^{l \pmod n})a_n = 2^{l+k-1-n}. \end{aligned} \tag{3.1.9}$$

We take an index l between $n-k+1$ and $n-t$. Because $t \leq k-1$, such an index exists and moreover is different from i, because $i \leq n-k$. Then, equation (3.1.7) becomes:

$$\begin{aligned} \alpha a_l + 2^{l+t-1}a_i + 2^{l-1}a_{n-t+1} + \\ +(1-\delta_{n-t+2})2^{l+t-2}a_{n-t+2} + \dots \\ \dots + (1-\delta_n)2^l a_n = 2^{l+k-1-n}. \end{aligned} \tag{3.1.10}$$

Multiplying (3.1.10) with $2^{n-t-l+1}$, subtracting (3.1.9) for $n-t+1$ and subdividing by α, we obtain:

$$2^{n-t-l+1}a_l + a_i = 0.$$

We recall that all a_j should be non-negative, so we get $a_i = 0$ and $a_l = 0$ for all $n-k+1 \leq l \leq n-t$.

On the other hand, subtracting from (3.1.8), equation (3.1.9) for $n-t+1$ multiplied by $2^{i+t-1-n}$ and subdividing by α, we obtain $a_i = 2^{k-t}$. Which is a contradiction. ∎

3.1.2.3 Subdivision of singular cones

We describe here how to subdivide the cones of the List $\mathcal{A}_k$ by help of the vector $2^k \star h_n$. First, we give an example.

Example 3.15. Subdivision process for $n = 4$. We first start by adding vector $h_4 = \frac{1}{15}(1, 2, 2^2, 2^3)$ to the simplex $\sigma_0 := \langle e_1, e_2, e_3, e_4 \rangle$. We subdivide into 4 cones by replacing in turn each of the e_i by h_4. We obtain one smooth 4−dimensional cone $\langle h_4, e_2, e_3, e_4 \rangle$, and three singular 4−dimensional cones $\langle e_1, e_2, e_3, h_4 \rangle$, $\langle e_1, h_4, e_3, e_4 \rangle$, $\langle e_1, e_2, h_4, e_4 \rangle$ (see also Figure 3.15). Thus, we obtain a partial desingularisation X_0 for the quotient $Y = \mathbb{A}^4/\mu_{15}$. It is a divisorial contraction, because the corresponding subdivision of the fan occurs by help of one vector.

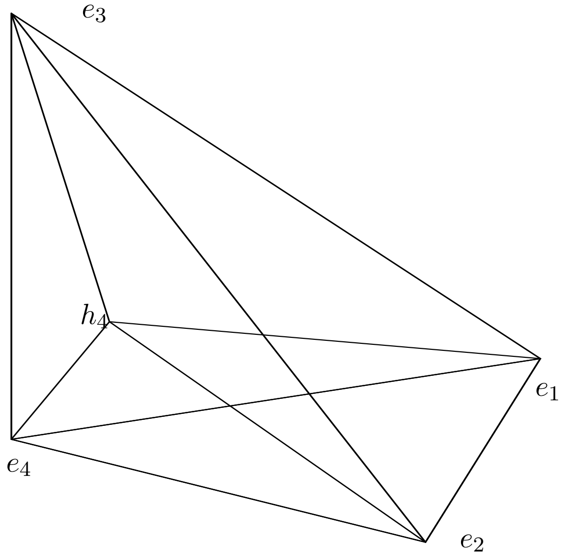

Figure 3.2: Subdivision at Step 0.

We consider now the vector $2 \star h_4$ and the only $4-$dimensional singular cone form the above list that contain it, namely $\langle e_1, e_2, e_3, h_4\rangle$. The vector $2 \star h_4$ is a primitive vector in the cone $\langle e_1, e_2, e_3, h_4\rangle$ (by Lemma 3.4). The subdivision process says that at this step we subdivide the cone $\langle e_1, e_2, e_3, h_4\rangle$ and keep all other cones unchanged. We obtain two $4-$dimensional smooth cones, $\langle 2\star h_4, e_2, e_3, h_4\rangle$ and $\langle e_1, e_2, e_3, 2\star h_4\rangle$, and two $4-$dimensional singular ones $\langle e_1, 2\star h_4, e_3, h_4\rangle$, $\langle e_1, e_2, 2\star h_4, h_4\rangle$ (see Figure 3.3). This gives a new partial resolution X_1 with a morphism $f_1 : X_1 \to X_0$. This is a divisorial contraction.

At this step, we have a total of three smooth and four singular $4-$dimensional cones, namely $\langle h_4, e_2, e_3, e_4\rangle$; $\langle 2 \star h_4, e_2, e_3, h_4\rangle$ and $\langle e_1, e_2, e_3, 2 \star h_4\rangle$; respectively $\langle e_1, h_4, e_3, e_4\rangle$, $\langle e_1, e_2, h_4, e_4\rangle$ and $\langle e_1, 2 \star h_4, e_3, h_4\rangle$, $\langle e_1, e_2, 2 \star h_4, h_4\rangle$.

We consider the vector $2^2\star h_4$. There are two singular cones containing it, namely $\langle e_1, e_2, h_4, e_4\rangle$ and $\langle e_1, e_2, 2\star h_4, h_4\rangle$. We first subdivide $\langle e_1, e_2, h_4, e_4\rangle$. Remark that there are two new smooth $4-$dimensional cones $\langle 2^2\star h_4, e_2, h_4, e_4\rangle$ and $\langle e_1, e_2, h_4, 2^2\star h_4\rangle$ and one singular $4-$dimensional cone $\langle e_1, 2^2\star h_4, h_4, e_4\rangle$.

Finally, we subdivide the cone $\langle e_1, e_2, 2 \star h_4, h_4\rangle$. The resulting cones are two smooth $\langle 2^2 \star h_4, e_2, 2 \star h_4, h_4\rangle$, $\langle e_1, e_2, 2^2 \star h_4, 2 \star h_4\rangle$ and one singular $\langle e_1, 2^2 \star h_4, 2 \star h_4, h_4\rangle$.

We consider the lattice $N = \mathbb{Z}^n + h_n\mathbb{Z}$ and the fan formed with:

• smooth cones

$$\langle h_4, e_2, e_3, e_4\rangle,$$

$$\langle 2 \star h_4, e_2, e_3, h_4\rangle,$$

$$\langle e_1, e_2, e_3, 2 \star h_4 \rangle,$$

$$\langle 2^2 \star h_4, e_2, h_4, e_4 \rangle,$$

$$\langle e_1, e_2, h_4, 2^2 \star h_4 \rangle,$$

- singular cones

$$\langle e_1, h_4, e_3, e_4 \rangle,$$

$$\langle e_1, 2 \star h_4, e_3, h_4 \rangle,$$

$$\langle e_1, 2^2 \star h_4, e_4, h_4 \rangle,$$

$$\langle e_1, 2^2 \star h_4, 2 \star h_4, h_4 \rangle.$$

Thus, we obtain a partial resolution $f_2 : X_2 \to X_1$, with f_2 divisorial contraction.

Now, we proceed to the last subdivision step using $2^3 \star h_4$. We start by subdividing the cone $\langle e_1, h_4, e_3, e_4 \rangle$, then we continue with $\langle e_1, 2 \star h_4, e_3, h_4 \rangle$ and $\langle e_1, 2^2 \star h_4, h_4, e_4 \rangle$, and we end with $\langle e_1, 2^2 \star h_n, 2 \star h_4, h_4 \rangle$.

At the end of this step, all the cones we obtain are smooth and the variety X_4 thus constructed is nothing else but $\mu_{15} - \mathrm{Hilb}\mathbb{A}^4$. Compare also the list of smooth cones with the one obtained in Chapter 1. We also have a chain of partial resolutions:

$$X_3 = \mu_{15} - \mathrm{Hilb}\mathbb{A}^4 \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} Y = \mathbb{A}^4/\mu_{15}. \clubsuit$$

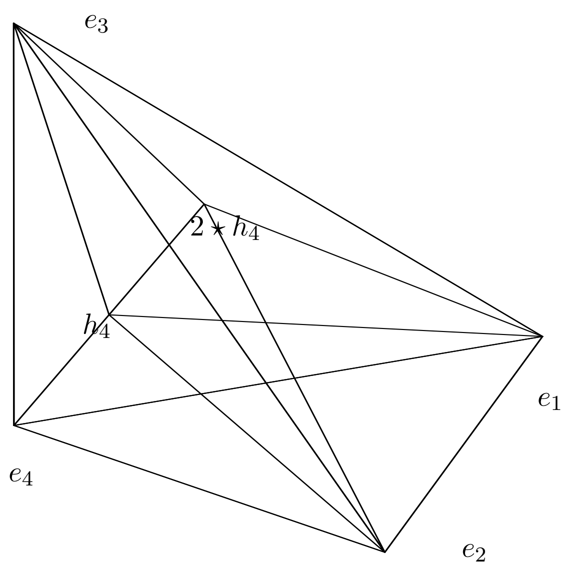

Figure 3.3: Subdivision at Step 1.

Proposition 3.16. *Let γ be a cone of list $\mathcal{A}_k$. By help of vector $2^k \star h_n$ we can subdivide γ into two $n-$dimensional smooth cones and $n-k-1$ singular $n-$dimensional cones.*

Proof:

Such a cone is of the form $\gamma = \langle e_1, \dots, e_{n-k}, h_n, w_{n-k+2}, \dots, w_n\rangle$, for some choice of w_j either e_j or $2^{n-j+1} \star h_n$, $n-k+2 \leq j \leq n$. By Lemma 1.48, Chapter 1, replacing e_1, respectively h_n, by $2^k \star h_n$ in such a cone, provides two smooth cones. By Lemma 1.52, Chapter 1 such a cone is $n-$dimensional. By Lemma 3.4, replacing any of the $w_j, n-k+2 \leq j \leq n$, provides no $n-$dimensional cone. We can replace one of the vectors $e_2, \dots, e_{n-k}$, in turn, by $2^k \star h_n$. The resulting cones are singular by Lemma 1.48 and $n-$dimensional by Corollary 3.12. There are $n-k-1$ such cones. ∎

Corollary 3.17. *The cones obtained by subdivision process by help of the vector $2^k \star h_n$ are in the List $\mathcal{C}_k$.*

Proof:

We want to describe the sets $\mathcal{NR}_k$ and $\mathcal{NS}_k$. For this, let γ be one of the cones $\sigma_{k,t} := \langle e_1, \dots, e_{n-k}, h_n, e_{n-k+2}, \dots, e_{n-t}, 2^t \star h_n, w_{n-t+2}, \dots, w_n\rangle$ of the list $\mathcal{A}_k$, for some positive $t \leq k-1$. The smooth cones of $\mathcal{NR}_k$ are obtained by replacing in $\gamma = \sigma_{k,t}$, with $1 \leq t \leq k-1$, the vector h_n (respectively e_1) by $2^k \star h_n$. We obtain smooth $n-$dimensional cones of the form

$$\langle 2^k \star h_n, e_2, \dots, e_{n-k}, h_n, e_{n-k+2}, \dots, e_{n-t}, 2^t \star h_n, w_{n-t+2}, \dots, w_n\rangle,$$

respectively

$$\langle e_1, \dots, e_{n-k}, 2^k \star h_n, e_{n-k+2}, \dots, e_{n-t}, 2^t \star h_n, w_{n-t+2}, \dots, w_n\rangle.$$

This agrees with Lemma 1.48, Section 1.3, Chapter 1. We deduce that $\#\mathcal{NR}_k = 2^{k-2}$.

The singular cones obtained from $\gamma = \sigma_{k,t}$ above are the cones in which we replace in turn one of the vectors $e_2, \dots, e_{n-k}$ by the vector $2^k \star h_n$. Thus, the set $\mathcal{NS}_k$ is formed with the cones:

$$\langle e_1, \dots, e_{j-1}, 2^k \star h_n, e_{j+1}, \dots, e_{n-k}, h_n, e_{n-k+2}, \dots, e_{n-t}, 2^t \star h_n, w_{n-t+2}, \dots, w_n\rangle,$$

$2 \leq j \leq n-k$ and some positive t.

Now, if we replace in a cone $\langle e_1, \dots, e_{n-k}, h_n, e_{n-k+2}, \dots, e_{n-t}, 2^t \star h_n, w_{n-t+2}, \dots, w_n\rangle$ one of the w_j by $2^k \star h_n$, we obtain no $n-$dimensional cone, because the common face $\langle e_1, \dots, e_{n-k}, h_n\rangle$ already contains the vector $2^k \star h_n$ (cf. Lemma 3.4). So, we don't consider such a cone. ∎

Remark 3.18. The above construction can be seen more geometrically as follows. By Lemma 3.14, any cone of $\mathcal{A}_k$ has the subcone $\langle e_1, \ldots, e_{n-k}, h_n\rangle$ as a face. The vector $2^k \star h_n$ is inside this face and provides a barycentric subdivision of it (see Lemma 3.4). Thus, the cone $\langle e_1, \ldots, e_{n-k}, h_n\rangle$ can be partitioned into $n-k+1$ faces τ_i, $1 \leq i \leq n-k+1$. Each such cone is obtained by replacing in turn one of the vectors $e_1, \ldots, e_{n-s}$ or h_n by $2^s \star h_n$. Any cone τ_i, $1 \leq i \leq n-k+1$, has dimension $n-k+1$. That is because the vectors generating a cone τ_i are part of a family of n linearly independent vectors, according to Corollary 3.12.

Now, for an index $j \geq n-k+2$, the vectors e_j and $2^{n-j+1} \star h_n$ lay on opposite sides of $\langle e_1, \ldots, e_{n-k}, h_n\rangle$. Let us fix $w_{n-k+2}, \ldots, w_n$, with w_j, $n-k+2 \leq j \leq n$ either e_j or $2^{n-j+1} \star h_n$. We can thus consider all the $n-$dimensional (cf. Corollary 3.12) cones $\langle \tau_i, w_{n-k+2}, \ldots, w_n\rangle$, where w_j, $n-k+2 \leq j \leq n$ is either e_j or $2^{n-j+1} \star h_n$, $1 \leq i \leq n-k+1$, fixed. We obtain a total of $n-k+1$ cones of dimension n. Among those, we get two smooth and $n-k-1$ singular $n-$dimensional new cones, as before. ♣

Remark 3.19.

1. Recurrently, we obtain a total of $1+1+2+2^2+\cdots+2^{k-2} = 2^{k-1}$ cones in which $2^k \star h_n$ occurs, which is in accordance with the previous results.
2. We remark that none of the cones of the List $\mathcal{A}_k$ contains the vector e_{n-k+1}. ♣

3.1.2.4 Step k and the end of the induction

We end here the proof of Theorem 3.5.

Proposition 3.20. *At the end of Step k, we obtain all the cones of the List $\mathcal{C}_k$. Moreover, any two such cones do not overlap.*

Proof:

Following the induction hypothesis we notice that $\mathcal{C}_k$ is contained in $\mathcal{D}_k := (\mathcal{C}_{k-1} \setminus \mathcal{A}_k) \cup \mathcal{NS}_k \cup \mathcal{NR}_k$. All the cones of $\mathcal{D}_k$ have the required form of List $\mathcal{C}_k$. To end the proof, it is enough to show that the cones of the set $\mathcal{D}_k$ do not overlap. Thus we obtain the equality $\mathcal{C}_k = \mathcal{D}_k$ and the non-overlapping property.

For this, let σ and σ' be two cones of $\mathcal{D}_k$. If they are both in $\mathcal{C}_{k-1} \setminus \mathcal{A}_k$ the property follows from the induction hypothesis for $\mathcal{C}_{k-1}$. If one cone — say σ' — is in $\mathcal{C}_{k-1} \setminus \mathcal{A}_k$ and the other one is in $\mathcal{NS}_k \cup \mathcal{NR}_k$, then the property follows also from the induction hypothesis. That comes from the subdivision process: the cone σ is a convex combination of some vectors contained in a cone τ of $\mathcal{A}_k$. By recurrence, τ and σ' do not overlap, so neither do σ and σ'.

To end the proof, let us see that is both σ and σ' are in $\mathcal{NS}_k \cup \mathcal{NR}_k$, then, they do not overlap. If σ and σ' are both smooth, the description of smooth cones implies that they do not overlap (see also Lemma 1.48, Section 1.3, Chapter 1). Now, if σ is smooth and σ' is singular, it is clear that σ' can not be contained in σ. The proof for the fact that σ' can not contain σ is similar to the one bellow, so we skip it.

So, we can suppose that both σ and σ' are singular cones of the set $\mathcal{NR}_k$. Thus, we can suppose that the cones σ and σ' are of the form:

$$\sigma := \langle e_1, \dots, e_{i-1}, h_n, e_{i+1}, \dots, e_{n-k}, 2^k \star h_n, w_{n-k+2}, \dots, w_n \rangle,$$

with $2 \leq i \leq n-k-1$ and each w_j, either e_j or $2^{n-j+1} \star h_n$ and

$$\sigma' = \langle e_1, \dots, e_{i'-1}, h_n, e_{i'+1}, \dots, e_{n-k}, 2^k \star h_n, w'_{n-k+2}, \dots, w'_n \rangle,$$

with $1 \leq i' \leq n-k-1$ and some choice of w'_j, either e_j or $2^{n-j+1} \star h_n$.

In particular, by Lemma 3.4, the vector $2^{n-i+1} \star h_n$ is in σ. If $i' < i$, then, the same proof as for Lemma 3.14, shows that vector $2^{n-i+1} \star h_n$ can not be in σ'. If $i' > i$, then $e_{i'}$ is in σ, so should also be in σ'. But this would imply that the $n-$dimensional cone $\langle e_1, \dots, e_{n-k}, h_n, w'_{n-k+2}, \dots, w'_n \rangle$ would be in σ' which is impossible (a subdivided cone would contain the cone it comes from). Thus, we conclude that $i = i'$.

Consider now the first index $j \geq n-k+2$ for which $w_j \neq w'_j$. Then, for making a choice, suppose that $w_j = e_j$ and $w'_j = 2^{n-j+1} \star h_n$. As before, this implies that vector e_j is a linear combination with non-negative coefficients of the vectors that form σ'. In particular, using the positiveness and the i^{th} line of the system, we get $a_i = a_j = a_{n-k+1} = 0$. Then, replacing in line j we get – here as usually δ_l is one if e_l occurs in τ, zero otherwise:

$$2^{j-n+k-2}(1-\delta_{n-k+2})a_{n-k+2} + \cdots + 2(1+\delta_{j-1})a_{j-1} = -1.$$

This is a contradiction because right hand side is positive and left hand side is negative. Which ends the proof. ∎

To conclude this section, at the end of Step k, we have constructed a toric variety X_k of lattice $N = \mathbb{Z}^n + h_n\mathbb{Z}$ and fan formed with the cones obtained following a subdivision process by help of vector $2^k \star h_n$, that is the cones of the set $\mathcal{C}_k$. We have a toric map $f_k : X_k \to X_{k-1}$. The vector $2^k \star h_n$ provides a divisor. This divisor forms the exceptional locus of the map f_k, which is thus a divisorial contraction.

3.2 The technical machinery

So far, for any index k, $0 \leq k \leq n-1$, we provided a toric variety X_k and we have a chain of partial resolutions (divisorial contractions):

$$X_{n-1} \xrightarrow{f_{n-1}} X_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} X_0 \xrightarrow{f_0} Y = \mathbb{A}^n/\mu_{2^n-1}. \tag{3.2.1}$$

We want to apply some technical machinery in order to prove Theorem 3.1. This section has the following structure. First we state Kawamata's theorem. Then, we give a collection of technical results, more or less known in the folklore of the derived categories, but not necessarily written explicitly before. We end by a recall of Kawamata's result, with application to our case.

3.2.1 Kawamata's result

In the sequel, notations of Section 2.4, Chapter 2 hold. In the section, we state Kawamata's result ([22] Theorem 4.2) giving sufficient conditions for a divisorial contraction to provide a Fourier-Mukay transform which is an equivalence at stack level. The terms in which he constructs his proof are much too general for our purpose, mainly because his result uses log pairs, while this is not at all necessary to our case, where the divisors $B = C = 0$. We prefer to quote here Kawamata's result in all its generality.

Theorem 3.21. *([22], Theorem 4.2, (2)) Let $f : (X, B) \to (Y, C)$ be a toroidal divisorial contraction between quasi-smooth toroidal varieties with effective $\mathbb{Q}-$divisors, whose coefficients are in $\{1 - 1/r \mid r \in \mathbb{N}\}$ and such that moreover $C = f_*B$. Let $\mathcal{X}$, respectively $\mathcal{Y}$ denote the smooth Deligne-Mumford stacks associated to X, respectively Y and denote by $\mathcal{W}$ the normalization of the fiber product $\mathcal{X} \times_Y \mathcal{Y}$, with morphisms $p : \mathcal{W} \to \mathcal{X}$, $q : \mathcal{W} \to \mathcal{Y}$.*

Suppose that the following holds:

$$K_X + B \geq f^*(K_Y + C). \tag{3.2.2}$$

Then, the Fourier-Mukai type functor

$$Rp_*Lq^* : D^b(Coh(\mathcal{Y})) \to D^b(Coh(\mathcal{X}))$$

is fully faithful. Moreover, if (3.2.2) is an equality, the above functor is an equivalence of derived categories.

Then, because we use only the case $B = C = 0$, we stat the following more appropriate result we use in the sequel.

Corollary 3.22. *Let $f : X \to Y$ be a toroidal divisorial contraction between quasi-smooth toroidal varieties . Let $\mathcal{X}$, respectively $\mathcal{Y}$ denote the smooth*

Deligne-Mumford stacks associated to X, respectively Y and denote by $\mathcal{W}$ the normalization of the fiber product $\mathcal{X} \times_Y \mathcal{Y}$, with morphisms $p : \mathcal{W} \to \mathcal{X}$, $q : \mathcal{W} \to \mathcal{Y}$.

Suppose that the following holds:

$$K_X \geq f^*(K_Y). \tag{3.2.3}$$

Then, the Fourier-Mukai type functor

$$Rp_*Lq^* : D^b(Coh(\mathcal{Y})) \to D^b(Coh(\mathcal{X}))$$

is fully faithful. Moreover, if (3.2.3) is an equality, the above functor is an equivalence of derived categories.

We remark also that this theorem can not be directly applied for the case when Y is the quotient $\mathbb{A}^n/\mu_{2^n-1}$ and X is the $\mu_{2^n-1}-$Hilbert scheme of $\mathbb{A}^n$ because the resolution morphism is not a divisorial contraction. Thus the need of the following section.

3.2.2 Prerequisites

Before starting, we state some results relating the theory of derived categories with the algebraic geometry. Some of the results are not new, but as far as the author knows there is no explicit proof in the literature. We sketch some of them.

The section is organized as follows. The first part contains a collection of results on equivalences of derived categories of coherent sheaves on smooth varieties. There are some applications of the Bondal-Orlov criteria. The second part states the similar results for the categories of derived categories of coherent sheaves on smooth Deligne-Mumford stacks. Here, the main tool are the point-sheaves introduced by Kawamata.

3.2.2.1 On bounded derived categories of coherent sheaves on smooth varieties

In the sequel we fix a base scheme $S = \mathrm{Spec}\mathbb{C}$. We denote by $X, Y, Z, \ldots$ the varieties on S. The fiber product of two such varieties X and Y over S is denoted by $X \times Y$. The bounded derived category of coherent sheaves on a variety X is denoted by $D^b(X)$. For a definition of this notion, see for example the first chapter of [21].

To provide an equivalence between derived categories

$$F : D^b(Y) \to D^b(X)$$

one needs first a good control of the underlying categories and secondly good techniques to check that a functor is an equivalence.

The first requirement can be satisfied by providing a class of objects "generating" the derived category. Thus the following definition.

Definition 3.23. *([6], Definition 2.1)A class Ω of objects of a triangulated category $\mathcal{A}$ is a spanning class for $\mathcal{A}$ if for any object a of $\mathcal{A}$ the following two conditions hold:*

$$\mathrm{Hom}_{\mathcal{A}}(\omega, a[i]) = 0, \forall \omega \in \Omega, \forall i \in \mathbb{Z} \Rightarrow a \simeq 0,$$
$$\mathrm{Hom}_{\mathcal{A}}(a[i], \omega) = 0, \forall \omega \in \Omega, \forall i \in \mathbb{Z} \Rightarrow a \simeq 0,$$

Remark 3.24. According to [6], Example 2.2 , for a smooth projective variety X, the set of skyscraper sheaves $\{\mathcal{O}_x, x \in X\}$ is a spanning class. Remark that a skyscraper sheaf has zero-dimensional support. ♣

Using this tool, one can state in the theory of triangulated categories the following criteria:

Theorem 3.25. *([6], Theorem 2.3) Let $\mathcal{A}$ an $\mathcal{B}$ be two triangulated categories and $F : \mathcal{A} \to \mathcal{B}$ an exact functor with left and right adjoint. Then, F is fully faithful if and only if there exists a spanning class Ω of $\mathcal{A}$, such that for any ω, ω' in Ω and any integer i the homomorphism*

$$F : \mathrm{Hom}_{\mathcal{A}}(\omega, \omega'[i]) \to \mathrm{Hom}^i_{\mathcal{B}}(F\omega, F\omega'[i])$$

is an isomorphism.

This criteria is very useful for the case of bounded derived categories of coherent sheaves on smooth varieties. An important class of functors for which this theory can be applied is the class of Fourier-Mukai transforms.

Definition 3.26. *([39], Definition 3.1.1) Let X and Y be two smooth projective varieties over a base scheme S and $p : X \times Y \to X$, respectively $q : X \times Y \to Y$ the two projection. Let K be an object of $D^b(X \times Y)$. A kernel-functor (Fourier-Mukai functor, integral functor) is a functor $\Phi_K : D^b(Y) \to D^b(X)$ defined by:*

$$\Phi_K(\bullet) = Rp_*(K \otimes^L q^* \bullet) \tag{3.2.4}$$

Remark 3.27. By [6], Lemma 4.5, it is known that such a functor admits a left and right adjoint. ♣

One main result on Fourier-Mukai transforms is the following

Theorem 3.28. *([39], Proposition 3.2)* **Bondal-Orlov criteria** *A Fourier-Mukai functor Φ_K as above is fully faithful if and only if, for all points y, y' of Y we have:*

$$\mathrm{Hom}(\Phi_K(\mathcal{O}_y), \Phi_K(\mathcal{O}_{y'})[i]) = \begin{cases} 0 & \text{except if } y = y' \text{ and } 0 \leq i \leq \dim Y \\ \mathbb{C} & \text{if } y = y' \text{ and } i = 0. \end{cases} \tag{3.2.5}$$

Using these results, we can state the following:

Proposition 3.29. *Let X and Y be as in Definition 3.26 and let K be an object of $D^b(X \times Y)$. We assume that X and Y have a covering by open subsets $\{X_1, X_2\}$, respectively $\{Y_1, Y_2\}$, such that the following conditions on the supports holds:*

$$1) \text{ Supp } K|_{X \times Y_i} \subset X_i \times Y_i, \forall i = 1, 2;$$

$$2) \text{ Supp } K|_{X_i \times Y} \subset X_i \times Y_i, \forall i = 1, 2.$$

Then, Φ_K is fully faithful if, and only if, $\Phi_{K|_{X_i \times Y_i}} : D^b(Y_i) \to D^b(X_i)$ is fully faithful for $i = 1, 2$.

Idea of a proof:

The conditions on the support imply in particular that, for $i, j \in \{1, 2\}$, $i \neq j$, and for a point y_j that doesn't belong to Y_i, we have:

$$\text{Supp}\Phi_K(\mathcal{O}_{y_j}) \bigcap X_i = \emptyset.$$

Also, for any point y_i of Y_i, we have:

$$\text{Supp}\Phi_K(\mathcal{O}_{y_i}) \subset X_i.$$

Applying Bondal-Orlov criteria, it is enough to prove that (3.2.5) holds. Let us consider two points y and y' of Y. Two cases can occur.

• If both y and y' are in the same open set Y_i, then we have $\text{Supp}\Phi_K(\mathcal{O}_y) \subset X_i$ and also $\text{Supp}\Phi_K(\mathcal{O}_{y'}) \subset X_i$. For an integer k we have:

$$\text{Hom}_{D^b(X)}(\Phi_K(\mathcal{O}_y), \Phi_K(\mathcal{O}_{y'}[k]) \simeq$$
$$\text{Hom}_{D^b(X_i)}(\Phi_K(\mathcal{O}_y)|_{X_i}, \Phi_K(\mathcal{O}_{y'}[k]|_{X_i}).$$

Now, $\Phi_K(\mathcal{O}_y)|_{X_i}$ is isomorphic to $\Phi_{K|_{X_i \times Y_i}}(\mathcal{O}_y)$. We conclude that:

$$\text{Hom}_{D^b(X)}(\Phi_K(\mathcal{O}_y), \Phi_K(\mathcal{O}_{y'}[k]) \simeq$$
$$\text{Hom}_{D^b(X_i)}(\Phi_{K|_{X_i \times Y_i}}(\mathcal{O}_y), \Phi_{K|_{X_i \times Y_i}}(\mathcal{O}_{y'}[k]).$$

Thus, (3.2.5) follows from the fully faithfulness of $\Phi_{K|_{X_i \times Y_i}} : D^b(Y_i) \to D^b(X_i), i = 1, 2$.

• We consider the case when y in one open set, say Y_1 and y' doesn't belong to Y_1. Then, the conditions on the support imply that

$$\text{Hom}_{D^b(X)}(\Phi_K(\mathcal{O}_y), \Phi_K(\mathcal{O}_{y'})[k]) = 0.$$

We conclude that Bondal-Orlov criteria holds for $\Phi_K : D^b(Y) \to D^b(X)$, which ends the proof. ∎

3.2.2.2 On bounded derived categories of coherent sheaves on smooth Deligne-Mumford stacks

Some of the results of the previous section hold in a modified form if we replace smooth varieties by smooth Deligne-Mumford stacks, as defined in Section 2.4.2, Chapter 2. The idea is to define a similar notion of a sky-scraper sheaf, this time for stacks.

Remark 3.30. (cf. [23], Example 5.5) Let Y be a quasi-projective variety, with a global cover that is locally of the form $U \to U/G_U$, for some smooth variety U and some finite group G_U. Denote by $\mathcal{Y}$ its associated smooth Deligne-Mumford stack.

Let us fix some local cover $\pi : U \to U/G_U$ and we denote the group G_U by G. We consider a point x of U and denote by Gx its orbit.

If the stabilizer of x is not trivial, denote it by H, subgroup of G. Let Irr_H be the set of all irreducible characters of H. Any irreducible character ρ has an associated vector space V_ρ. We take all the $G-$equivariant sheaves on U

$$\mathcal{P}_{x,\rho,\pi} = \bigoplus_{y \in Gx} V_\rho \otimes_{\mathbb{C}} \mathcal{O}_y.$$

We also denote by $\mathcal{P}_{x,\rho,\pi}$ the corresponding sheaf on the stack $\mathcal{Y}$. According to [23], Example 5.5, the set of all those sheaves, while $\pi : U \to U/G_U$ local covering, x a point of U and ρ runs over all characters of the stabilizer, is a spanning class for the bounded derived category of coherent sheaves on the smooth stack $\mathcal{Y}$. ♣

Definition 3.31. *We call the sheaf $\mathcal{P}_{x,\rho,\pi}$ of the previous remark a point-sheaf.*

Kawamata shows that the class of all point-sheaves $\mathcal{P}_{x,\rho,\pi}$, while x runs over U and ρ in the set of irreducible characters, forms a spanning class for the bounded derived category of $G-$equivariant coherent sheaves on the smooth Deligne-Mumford stack $\mathcal{Y}$.

Remark 3.32. For the spanning class of point-sheaves, one can sate similar results as the one of Proposition 3.29. The proof uses a weak stack version of the Bondal-Orlov criteria based on point-sheaves notion and follows from Theorem 3.25.

Thus, let X and Y be two normal varieties with at most quotient singularities and let $f : X \to Y$ be a map between them. We denote by:

1. $\mathcal{X}$, respectively $\mathcal{Y}$, the associated smooth Deligne-Mumford stacks for X, respectively Y,

2. F the corresponding map at stack level between $\mathcal{X}$ and $\mathcal{Y}$,

$$\begin{tikzcd}
 & (\mathcal{X} \times_Y \mathcal{Y})^{\sim} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathcal{X} \arrow[rr, "F"] \arrow[d] & & \mathcal{Y} \arrow[d] \\
X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

Figure 3.4: Diagram for Lemma 3.32.

3. p, respectively q, the projection from $(\mathcal{X} \times_Y \mathcal{Y})^{\sim}$, normalization of the fiber product $\mathcal{X} \times_Y \mathcal{Y}$, to $\mathcal{X}$, respectively $\mathcal{Y}$, as summarized in Figure 3.4

We suppose that X and Y have open coverings by open subvarieties $\{X_1, X_2\}$, respectively $\{Y_1, Y_2\}$, such that:

$$f(X_i) \subset Y_i \text{ and } f^{-1}(Y_i) \subset X_i, \text{ for } i = 1, 2.$$

We make the following notations:

1. $\mathcal{X}_i := \mathcal{X} \times_X X_i$, $\mathcal{Y}_i := \mathcal{Y} \times_Y Y_i$, the smooth Deligne-Mumford stacks associated to X_i, respectively Y_i, $i = 1, 2$,

2. p_i and q_i are the corresponding projections from the normalization of the fiber product:

$$\begin{tikzcd}
 & (\mathcal{X}_i \times_{Y_i} \mathcal{Y}_i)^{\sim} \arrow[dl, "p_i"'] \arrow[dr, "q_i"] & \\
\mathcal{X}_i & & \mathcal{Y}_i
\end{tikzcd}$$

Then,

$$Rp_*Lq^* : D^b(\mathcal{Y}) \to D^b(\mathcal{X}) \text{ is fully faithful if, and only if,}$$
$$Rp_{i*}Lq_i^* : D^b(\mathcal{Y}_i) \to D^b(\mathcal{X}_i) \text{ is fully faithful for } i = 1, 2.$$

♣

3.2.3 Résumé of Kawamata's proof

We give a brief description of Kawamata's proof for Corollary 3.22, stating the main steps as claims, with some hints for the proofs. In each subsequent remark after such a claim, we explain what should be the equivalent of the claim in our case, that is the case of partial resolutions between $\mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n$ and $\mathbb{A}^n/\mu_{2^n-1}$.

The aim is to understand the mechanism of the proof for Y an orbifold quotient singularity and for X a partial resolution, while the map f between them is a toric map.

Claim 3.33. *It suffices to prove that the functor of Corollary 3.22 is fully faithful.*

Idea of a proof:

Here, one can use the stack version of [6], Theorem 1.1, stating that a fully faithful functor is an equivalence of categories if and only if it commutes with the Serre functor. Under the hypothesis of equality in (3.2.3) this follows. ∎

In our case, both X and Y are toric varieties. In practice, at a first step, the variety Y is the quotient $\mathbb{A}^n/\mu_{2^n-1}$. Once we construct the partial resolution $X = X_0 \to Y = \mathbb{A}^n/\mu_{2^n-1}$, we apply again Corollary 3.16, this time for $Y = X_0$ and $X = X_1$, the newly constructed partial resolution for X_0. In the sequel, we fix once for all the lattice $N = \mathbb{Z}^n + h_n\mathbb{Z}$. At a first step, the fan of Y is the cone σ_0 generated by the vectors of a basis $\{e_1, \dots, e_n\}$. Then, for each variety X_k, $0 \leq k \leq n-1$, the cones of its fan are the cones of list $\mathcal{C}_k$.

Suppose that we are at Step k, $0 \leq k \leq n-1$, of the subdivision process. At this stage, we add the vector $2^k \star h_n$ and we subdivide each cone in the list $\mathcal{A}_k$ (and let the others unchanged), in order to provide the fan of the partial resolution X_k, as follows. By Lemma 3.4, the vector $2^k \star h_n$ can be expressed as a linear combination with positive coefficients of $n-k+1$ vectors. Following Kawamata's idea, we subdivide each cone of list $\mathcal{A}_k$, by replacing in turn one of its vectors by $2^k \star h_n$. By Proposition 3.16, each cone of list $\mathcal{A}_k$ splits into a number of $n-k+1$ cones of dimension n. Such a $n-$dimensional cone provides a copy of $\mathbb{A}^n$; we glue them together to recover the partial resolution X_k. The number of cones we added by the subdivision process agrees with the number of cones one should add by Kawamata's construction. By subdivision process, at each step k, we construct a map $f : X_k \to X_{k-1}$, a divisorial contraction.

Next, the idea is to take the "good" coverings for $X = X_k$ and $Y = X_{k-1}$ in order to obtain the corresponding smooth Deligne-Mumford stacks. Let $\sigma := \langle v_1, \dots, v_n \rangle$ be a cone of the list $\mathcal{C}_{k-1}$ and denote by $X_{k-1}(\sigma)$ its corresponding affine variety, seen as open set in X_{k-1}. We cover $X_{k-1}(\sigma)$ with one copy of $\mathbb{A}^n$ seen as toric variety with lattice generated by the vectors $v_1, \dots, v_n$ and fan formed with the cone σ. We can thus construct $\mathcal{X}_{k-1}$. The condition (2.4.1), Section 2.4.1, Chapter 2 holds because we can apply a similar result as the one of Lemma 3.37 bellow. Next, we want to find the corresponding covering for X_k. If the cone σ is in the list $\mathcal{A}_k$, during the subdivision process, we replace it by $n-k+1$ cones of dimension n, denoted σ_i, $1 \leq i \leq n-k+1$. We get new toric varieties $X_k(\sigma_i)$, that replace $X_{k-1}(\sigma)$. We take as a covering for each $X_k(\sigma_i)$ also a copy of $\mathbb{A}^n$. If the cone σ is not in $\mathcal{A}_k$, we keep $X_{k-1}(\sigma)$ and its covering unchanged. We thus construct $\mathcal{X}_k$. We have the following:

Claim 3.34. *The smooth Deligne-Mumford stacks associated to X_k, $0 \leq k \leq n-1$, are defined by the coverings with affine pieces $\mathbb{A}^n$.*

Claim 3.35. *([22], Lemma 4.3, 4.4) The class of invertible sheaves on $\mathcal{X}$ (respectively $\mathcal{Y}$) is a spanning class.*

Claim 3.36. *For any invertible sheaf L on $\mathcal{Y}$, the sheaf $Rp_*Lq^*(L)$ is invertible on $\mathcal{X}$.*

Using these claims, Kawamata proves that, the functor $Rp_*Lq^*(L)$ is fully faithful. We do not reproduce his proof here.

3.3 Proof of Theorem 3.1

Lemma 3.37. *Let X be a toric variety of lattice N and fan Δ and let Δ' be a subdivision of Δ We assume that N, each cone of Δ and each cone of Δ' are generated by junior points (see Notation 1.15). Let X' denote the toric variety of lattice N and fan Δ'. Then, the toric morphism $f : X' \to X$ is a proper, birational, crepant morphism.*

Proof:

The morphism f is proper and birational by [15], 2.6. For the crepancy, notice first that the vector $t := -(1, 1, \dots, 1)$ is in M, the dual of N. Let $\mathrm{div}(\chi^t)$ denote the Cartier divisor associated to the function χ^t. By [15], 3.3, the associated Weil divisor on X is $[\mathrm{div}(\chi^t)] = \sum_{\rho \in \Delta} \langle f(\rho), t \rangle D_\rho$, where we use the notations of Section 1.1.1, page 4. By our assumption, we get $\langle f(\rho), t \rangle = -1$. By [34], Corollary 3.3 and the subsequent Remark, we conclude that the canonical divisor of X is trivial. A similar result holds for X'. We conclude for the crepancy. ∎

Corollary 3.38. *Any divisorial contraction $X_k \to X_{k-1}$ of the subdivision process is crepant.*

Proof:

By Lemma 3.4, the fan of each variety X_{k-1} is subdivided by the vector $2^k \star h_n$ with coordinates $(\frac{1}{2^{n-k}}, \frac{1}{2^{n-k}}, \frac{1}{2^{n-k+1}}, \dots, \frac{1}{2})$, in the cone $\langle h_n, e_1, \dots, e_{n-k} \rangle$. ∎

Remark 3.39. In particular, this means that for any divisorial contraction $X_k \to X_{k-1}$ we are in the hypothesis of Corollary 3.22. Condition (3.2.3) holds with equality, thus we obtain an equivalence of derived categories $D^b(\mathcal{X}_k) \simeq D^b(\mathcal{X}_{k-1})$. ♣

Proof of Theorem 3.1

We consider the chain of partial resolutions provided by (3.2.1) and apply Corollary 3.22. This gives an equivalence of derived categories for each partial resolution, that is we have for any index $k, 0 \leq k \leq n-1$ an equivalence:

$$D^b(\mathcal{X}_k) \simeq D^b(\mathcal{X}_{k-1}).$$

Here, $\mathcal{X}_k$ is the smooth Deligne-Mumford stack associated to the partial resolution X_k. We made the convention that X_{-1} is the orbifold quotient $\mathbb{A}^n/\mu_{2^n-1}$. As proved in Corollary 3.8, at the last step we get that X_{n-1} is the $\mu_{2^n-1}-$Hilbert scheme of $\mathbb{A}^n$.

Let us denote by $[\mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n]$ the smooth Deligne-Mumford stacks associated to the $\mu_{2^n-1}-$Hilbert scheme of $\mathbb{A}^n$ and by $[\mathbb{A}^n/\mu_{2^n-1}]$ the smooth Deligne-Mumford stack associated to the orbifold $\mathbb{A}^n/\mu_{2^n-1}$. The chain of previous equivalences provides the derived equivalence:

$$D^b([\mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n]) \simeq D^b([\mathbb{A}^n/\mu_{2^n-1}]).$$

Because $\mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n$ is smooth, we see that:

$$D^b([\mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n]) \simeq D^b(\mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n).$$

By Example 2.90, Chapter 2, we have that:

$$D^b([\mathbb{A}^n/\mu_{2^n-1}]) \simeq D^b_{\mu_{2^n-1}}(\mathbb{A}^n).$$

Here, $D^b_{\mu_{2^n-1}}(\mathbb{A}^n)$ is the bounded derived category of coherent $\mu_{2^n-1}-$equivariant sheaves on the affine space $\mathbb{A}^n$.

We conclude that there exists a derived equivalence:

$$D^b(\mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n) \simeq D^b_{\mu_{2^n-1}}(\mathbb{A}^n).$$

■

Remark 3.40. It would be interesting to find a direct definition of the functor F providing the derived equivalence

$$D^b(\mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n) \simeq D^b_{\mu_{2^n-1}}(\mathbb{A}^n).$$

Using [7], page 537, one would expect a description in terms of universal closed subscheme $Z \subset \mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n \times \mathbb{A}^n$. More precisely, we have the commutative diagram:

$$\begin{tikzcd}
 & \mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n \times \mathbb{A}^n \arrow[ddl, "p"'] \arrow[ddr, "q"] & \\
 & Z \arrow[u, hook] \arrow[dl, "a"] \arrow[dr, "b"'] & \\
\mu_{2^n-1} - \mathrm{Hilb}\mathbb{A}^n \arrow[dr] & & \mathbb{A}^n \arrow[dl] \\
 & \mathbb{A}^n/\mu_{2^n-1} &
\end{tikzcd}$$

Then, the functor F should be the Fourier-Mukai transform of kernel $\mathcal{O}_Z$, that is $\mathbf{R}q_*(\mathcal{O}_Z \otimes p^*(\bullet \otimes \rho_0))$, where ρ_0 is the trivial representation of the group μ_{2^n-1}. ♣

Chapter 4

Conclusions

4.1 McKay correspondence

The result of Theorem 3.1, Chapter 3 is known as the McKay correspondence for derived categories. The result and the subsequent proof remain true if we take instead of the field of complex numbers any algebraically closed field κ of characteristic p prime with the order of the group $H_n := \mu_{2^n-1}$, that is $(p, 2^n - 1) = 1$.

4.2 Broué's conjecture

Let G be a finite group, p a prime number and κ an algebraically closed field of characteristic p.

The decomposition of the unity of κG into a sum of orthogonal primitive central idempotents $1 = \sum e$, corresponds to the decomposition of the algebra κG into a direct sum of indecomposable two-sided ideals $\kappa G = \bigoplus_e \kappa Ge$. Each such ideal is called a block of κG.

The augmentation map $\kappa G \to \kappa$ factorizes through a unique block of κG; we call this block the principal block of κG, denoted $B_0(\kappa G)$.

Then, Broué's conjecture is the following.

Conjecture 4.1. *If κ is an algebraically closed field of characteristic $p > 0$ and G a finite group with an abelian Sylow $p-$subgroup P, then, the principal block of κG is derived equivalent to the principal block of $\kappa N_G(P)$. Here, $N_G(P)$ denotes the normalizer of P in G.*

This conjecture was proved true for the group $\mathrm{SL}_2(\mathbb{F}_{p^n})$ and any prime characteristic p of κ, where $\mathbb{F}_{p^n}$ is the subfield of κ with p^n elements. The proof is given in [35]. Remark that this is not a graded version, even if $\kappa[\kappa^n \rtimes G]$ is naturally graded.

Consider now the case when κ is an algebraically closed field of characteristic two. The group $G := \mathrm{SL}_2(\mathbb{F}_{2^n})$ has a Sylow $2-$subgroup P which is:

$$P := \left\{ \left(\begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) \;\Big|\; u \in \mathbb{F}_{2^n} \right\}.$$

Let us denote by E the group of diagonal matrices:

$$E := \left\{ \left(\begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array} \right) \;\Big|\; x \in \mathbb{F}_{2^n}^* \right\}.$$

The normalizer of the group P is the group $N_G(P) = P \rtimes E$. The diagonal group E acts on P as follows. If g is an element of E, that is a matrix $\left(\begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array} \right)$, $a \in \mathbb{F}_{2^n}^*$ and h is an element of P, that is a matrix $\left(\begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right)$, $u \in \mathbb{F}_{2^n}$, then we have a diagonal action:

$$ghg^{-1} = \left(\begin{array}{cc} 1 & ux^2 \\ 0 & 1 \end{array} \right).$$

The action of E on P can be seen as an endomorphism of $\mathbb{F}_{2^n}$, preserving the additive structure of the field. That is an endomorphism on $\mathbb{F}_{2^n}$ as $\mathbb{F}_2$ vector space.

$$\left\{ \begin{array}{ccc} \mathbb{F}_{2^n} & \longrightarrow & \mathbb{F}_{2^n} \\ x & \longmapsto & ux^2 \end{array} \right.$$

In particular, the above endomorphism gives an endomorphism of the vector space $V := \mathbb{F}_{2^n} \otimes_{\mathbb{F}_2} \kappa$.

We are in the following situation. We have an $n-$dimensional vector space V over κ, with an action of the diagonal group G. For a good basis of V, we get that this action is the same as the action of the group H_n on V.

4.3 A geometrical realization of Broué's conjecture

We fix ϵ a primitive root of unity of order $2^n - 1$. Let H_n the group of primitive roots of unity of order $2^n - 1$, seen as a subgroup of $\mathrm{SL}_2(\kappa)$ with generator the diagonal matrix $\mathrm{diag}(\epsilon, \epsilon^2, \ldots, \epsilon^{2^{n-1}})$. Let H_n act on the affine space $\mathbb{A}^n_\kappa$ by multiplication and consider also the scalar action of the multiplicative group κ^* on the vector space κ^n.

Theorem 3.1 extends to the κ^*-equivariant setting and gives an equivalence:

$$D^b_{\kappa^*}(H_n - \mathrm{Hilb}\mathbb{A}^n_\kappa) \simeq D^b_{\kappa^* \times H_n}(\mathbb{A}^n_\kappa). \tag{4.3.1}$$

Because of the action of κ^*, this last derived category is nothing else but the bounded derived category of H_n-equivariant finitely generated graded modules over the polynomial algebra in n variables, i.e. $D^b(S(\kappa^n) \rtimes H_n -$ grad).

The exterior algebra on κ^n, that is $\Lambda(\kappa^n)$, becomes a commutative algebra. If $\{v_1, \dots, v_n\}$ is a $\kappa-$basis of κ^n, then, for each index $i, 1 \leq i \leq n$, we have $v_i^2 = 0, (1+v_i)^2 = 1$. So, put $X_i = v_i + 1$, to obtain that $\Lambda(\kappa^n)$ is nothing else but $\kappa[X_1, \dots, X_n]/\langle X_1^2 - 1, \dots, X_n^2 - 1\rangle$, this is the group algebra $\kappa[(\mathbb{Z}/2\mathbb{Z})^n]$. This has a natural grading where the elements of the group H_n are in degree zero and the v_i are in degree one. According to [24], Section 10.5, first lemma, we have the Koszul equivalence of derived categories:

$$D^b(S(\kappa^n) \rtimes H_n - \text{grad}) \simeq D^b(\Lambda(\kappa^n) \rtimes H_n - \text{grad}), \tag{4.3.2}$$

hence

$$D^b(S(\kappa^n) \rtimes H_n - \text{grad}) \simeq D^b(\kappa[(\mathbb{Z}/2\mathbb{Z})^n \rtimes H_n - \text{grad}]). \tag{4.3.3}$$

We use here the graded version of Broué's conjecture. This is given by R. Rouquier in [40]. He shows that there is a grading also on the principal block $B_0(\mathrm{SL}_2(\kappa))$ and that a graded derived equivalence holds, that is

$$D^b(B_0(\kappa[\mathrm{SL}_2(\mathbb{F}_{2^n})]) - \text{grad}) \simeq D^b(B_0(\kappa[\kappa^n \rtimes H_n]) - \text{grad}).$$

Thus, a geometric realization of Broué's conjecture via the McKay correspondence:

$$D^b_{\kappa^*}(G - \text{Hilb}\kappa^n) \simeq D^b(B_0(\kappa[\mathrm{SL}_2(\kappa^n)]) - \text{grad}).$$

Appendix A

Trihedral groups

This section completes Section 1.1.4. Its aim is to give an algorithm for constructing $G-$Hilbert scheme of $\mathbb{A}^3$ for G non-commutative group. Unfortunately, this is work on progress, with Professor M. Reid, based on the PhD thesis of R. Leng [28], so the proofs are rather an idea of a proof, while theorems and propositions give place to claims and examples. In the first part, we recall R. Leng's results on binary-dihedral groups. The next part treats the case of trihedral groups, as the natural generalization of the binary-dihedral case. We follow again results of R. Leng and try to generalize her work for the case of a general trihedral group. The aim of A.2 is to give some general lines towards a magma implementation of an algorithm for computing $T - \mathrm{Hilb}\mathbb{A}^3$, for any trihedral group T. The first steps for such an implementation are given in Section A.2.3 containing a Magma programs written together with G. Brown PhD in Warwick University.

A.1 An example: binary-dihedral groups

In the sequel, one treats the example of the binary dihedral group, following [28]. From this, some natural generalizations for the trihedral groups are possible, as stated in the next section.

The case of an abelian group uses the particular fact that all the irreducible representations are of dimension one. The case of a non-commutative group is more difficult from this point of view. Fist of all, one needs to define a notion corresponding to the one of a monomial being in a $G-$representation. Already for a two-dimensional representation, one should consider not only monomials, but polynomials, more precisely pairs of polynomials (because we are in dimension two).

We see a binary dihedral group $BD_{4k}, k \geq 1$ as a subgroup of $SL_2(\mathbb{C})$, via an inclusion $R : BD_{4k} \hookrightarrow SL_2(\mathbb{C})$, as in the Section 1.1.4.We recall that the generators are

$$\sigma \mapsto g = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \tau \mapsto h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where ϵ is a primitive root of unity of order $(2k)^{\text{th}}$. The inclusion R is actually a two-dimensional representation, called the regular representation of BD_{4k}.

We recall that the irreducible representations of BD_{4k} are one of the following:

• four one-dimensional irreducible representations:

$L_1, L_2 : \begin{cases} g \mapsto 1 \\ h \mapsto \pm 1 \end{cases}$

$L_3, L_4 : \begin{cases} g \mapsto -1 \\ h \mapsto \pm i^n \end{cases}$

• $k-1$ two-dimensional irreducible representations:

$$V_j : \begin{cases} g \mapsto \begin{pmatrix} \varepsilon^j & 0 \\ 0 & \varepsilon^{-j} \end{pmatrix} \\ h \mapsto \begin{pmatrix} 0 & 1 \\ (-1)^j & 0 \end{pmatrix} \end{cases} j = 1, \ldots, k-1.$$

Definition A.1. *[[28], Definition 2.1, 2.2]*

1. *We say that a polynomial P of $\mathbb{C}[X_1, X_2]$ belongs to a one-dimensional irreducible representation of BD_{4k} if:*

   $\begin{cases} g \cdot P = P \\ h \cdot P = \pm P, \end{cases}$ *for L_1, L_2;*

   $\begin{cases} g \cdot P = -P \\ h \cdot P = \pm i^k P, \end{cases}$ *for L_3, L_4.*

2. *We say that a pair of polynomials (P, Q) of $\mathbb{C}[X_1, X_2]$ belongs to a two-dimensional irreducible representation V_j of BD_{4k} if:*

   $\begin{cases} Q = h \cdot P \\ g \cdot (P, Q) = (\epsilon^j P, \epsilon^{-j} h \cdot P). \end{cases}$

3. *A set of polynomials $\Gamma \subset \mathbb{C}[X_1, X_2]$ is called a $BD_{4k}-$graph if it contains*

   (a) one polynomial belonging to each one-dimensional irreducible representations L_1, L_2, L_3, L_4 and

   (b) two pairs of polynomials belonging to each two-dimensional irreducible representations V_j.

4. *If Γ is a $BD_{4k}-$graph, we denote by $I(\Gamma)$ the ideal generated by $\mathbb{C}[X_1, X_2] \setminus \Gamma$.*

As in the abelian case, we want to characterize the $BD_{4k}-$graphs Γ such that the associated ideal $I := I(\Gamma)$ defines a $BD_{4k}-$cluster of support $\{0\}$ in the $BD_{4k}-$Hilbert scheme. This is equivalent to ask that $\mathbb{C}[X_1, X_2]/I$ is the regular representation of BD_{4k} and Γ is a $\mathbb{C}-$basis for the quotient $\mathbb{C}[X_1, X_2]/I$. Following [28], Lemma 2.5, 2.7, 2.9, we have:

Claim A.2. 1. Constant monomial 1 can always be chosen to be in a $BD_{4k}-$graph.

2. Either X_1X_2 or $X_1^{2k} - X_2^{2k}$ belong to a $BD_{4k}-$graph (because both are associated to the representation L_2).

3. Either $X_1^k - i^kX_2^k$ or $X_1X_2(X_1^k + i^kX_2^k)$ belong to a $BD_{4k}-$graph (corresponding to L_3).

4. Either $X_1^k + i^kX_2^k$ or $X_1X_2(X_1^k - i^kX_2^k)$ belong to a $BD_{4k}-$graph (corresponding to L_4).

A systematic discussion (see [28], Section 2.4) on the choice of the [pairs of] polynomials belonging to each representation shows that a list of possible $BD_{4k}-$graphs is as follows. Here, we put each monomial $X_1^aX_2^b$ on line b, column a and the notation $[\cdot]$ means that we don't take the monomial $\cdot$ in the graph.

• Type A:

$$\begin{array}{ll}
[\cdot] & \\
[X_2^{2k}] & \\
X_2^{2k-1} & \\
\dots & \\
[X_2^k] & \\
\dots & \\
X_2 & [\cdot] \\
1 & X_1 \ \dots \ [X_1^k] \ \dots \ X_1^{2k-1} \ [X_1^{2k}] \ [\cdot],
\end{array}$$

and $X_1^k \pm i^kX_2^k, X_1^{2k} - X_2^{2k}$ and $X_1^k \pm i^kX_2^k, X_1X_2$ instead of X_1^k, X_2^k;

• Type B: for l from 2 to $k-2$,

$$\begin{array}{llll}
[\cdot] & & & \\
X_2^{2k-l} & & & \\
\dots & & & \\
X_2^{l+1} & [\cdot] & & \\
X_2^l & X_1X_2^l & [\cdot] & \\
\dots & & & \\
X_2^2 & X_1X_2^2 & [\cdot] & \\
X_2 & X_1 & X_1^2X_2 \ \dots \ X_1^lX_2 \ [\cdot] & \\
1 & X_1 & X_1^2 \ \dots \ \ X_1^l \ X_1^{l+1} \ \dots \ X_1^{2k-l}[\cdot], &
\end{array}$$

with $X_1^k \pm i^kX_2^k$ instead of X_1^k, X_2^k;

- Type C:

$$\begin{array}{lllllll}
[X_2^{k+1}] & [X_1X_2^{k+1}] & & & & & \\
[X_2^{k}] & [X_1X_2^{k}] & & & & & \\
X_2^{k-1} & X_1X_2^{k-1} & & & & & \\
\ldots & & & & & & \\
X_2^2 & X_1X_2^2 & [\cdot] & & & & \\
X_2 & X_1X_2 & X_1^2X_2 \ldots & X_1^{k-1}X_2 & [X_1^kX_2] & [X_1^{k+1}X_2] \\
1 & X_1 & X_1^2 \ldots & X_1^{k-1} & [X_1^k] & [X_1^{k+1}]
\end{array}$$

with one of the following types of polynomials instead of X_i^k, X_i^{k+1}, $X_iX_j^k$, $X_iX_j^{k+1}$, for $i=1,2, j=1,2, i\neq j$:

either $X_1^k+i^kX_2^k, X_1(X_1^k+i^kX_2^k), X_2(X_1^k+i^kX_2^k), X_1X_2(X_1^k+i^kX_2^k)$,
or $X_1^k-i^kX_2^k, X_1(X_1^k-i^kX_2^k), X_2(X_1^k-i^kX_2^k), X_1X_2(X_1^k-i^kX_2^k)$,
or $X_1^k+i^kX_2^k, X_1(X_1^k+i^kX_2^k), X_2(X_1^k+i^kX_2^k), X_1^k-i^kX_2^k$,
or $X_1^k-i^kX_2^k, X_1(X_1^k-i^kX_2^k), X_2(X_1^k-i^kX_2^k), X_1^k+i^kX_2^k$.

Now, in order to recover the ideal corresponding to a $BD_{4k}-$graph, take each type A,B,C defined before and compute the syzygies. This gives an explicit description of a cover of $BD_{4k}-$Hilb$\mathbb{A}^2$ by smooth affine surfaces. For a complete description of the syzygies see Section 2.5 of [28]. We remark that this method of computing the syzygies is a common technique, that we met also in the abelian case (see [33], Section 4 or [10], Section 5).

Finally, we have the following theorem ([28] Theorems 2.4 and 2.10):

Theorem A.3. *Let BD_{4k} be the binary dihedral group of order $4k$. Then:*

1. *If an ideal I defines a $BD_{4k}-$cluster, then it has an associated $BD_{4k}-$graph of type A,B or C. This $BD_{4k}-$graph is not unique.*
2. *By computing the syzygies, one can construct $BD_{4k}-$Hilb$\mathbb{A}^2$ by giving an affine cover by smooth surfaces of $\mathbb{A}^4$.*

A.2 Trihedral groups

This section aims to give an answer to the question how should the $G-$Hilb$\mathbb{A}^3$ look like for a non-abelian subgroup G of $SL_3(\mathbb{C})$. Trying to generalize the binary dihedral case, the first idea that occurs is to take a trihedral group. Remind that a binary dihedral group is a semi-direct product of two abelian subgroups of $SL_2(\mathbb{C})$. A trihedral group T is the semi-direct product of a diagonal abelian subgroup A of $SL_3(\mathbb{C})$ and μ_3 seen as a subgroup of $SL_3(\mathbb{C})$ by sending ω the primitive cubic root of unity to the matrix $\tau=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$. In order to recover $G-$Hilb$\mathbb{A}^3$ for such a group, the tricky part is to give a good definition of $T-$graphs in terms of irreducible representations. We will focus here only on the case when the normal subgroup A of the trihedral group $T := A \rtimes \mu_3$ is such that $\#A - 1 \equiv 0 \pmod 3$. Note that the other possible case is $\#A \equiv 0 \pmod 3$ which we do not treat here.

Towards a possible generalization, we consider, in parallel with the technical claims, some examples. We treat the case of a trihedral group for which the cyclic group A has order 79. The group A is seen as a diagonal subgroup of $SL_3(\mathbb{C})$ with generator $g := \mathrm{diag}(\epsilon, \epsilon^{23}, \epsilon^{55})$, where ϵ is a primitive root of unity of order 79. According to Notation 1.4, A is the group $\frac{1}{79}(1, 23, 55)$. We put $G := \frac{1}{79}(1, 23, 55) \rtimes \mu_3$.

A.2.1 Representations and trihedral graphs

A first task is to describe the representations of a trihedral group. For the group G above, there are three one-dimensional irreducible representations, $L_i, i = 1, 2, 3$ given by sending g to 1 and τ respectively to ω^i. Also, there are $(\#A - 1)/3 = 22$ irreducible representations of dimension three. For defining such a representation, send τ to it-self and g in turn to a matrix of the form $\mathrm{diag}(\epsilon^a, \epsilon^b, \epsilon^b)$, where a triple $[a, b, c]$ is in the set bellow:

$R :=$ { [1,23,55],	[2, 46, 31],	[3, 69, 7],	[4, 13, 62],
[5, 36, 38],	[6, 59, 14],	[8, 26, 45],	[9, 49, 21],
[10, 72, 76],	[11, 16, 52],	[12, 39, 28],	[15, 29, 35],
[17, 75, 66],	[18, 19, 42],	[20, 65, 73],	[22, 32, 25],
[24, 78, 56],	[27, 68, 63],	[30, 58, 70],	[33, 48, 77],
[34, 71, 53],	[37, 61, 60],	[40, 51, 67],	[41, 74, 43],
[44, 64, 50],	[47, 54, 57] }.		

Here one doesn't take the representation $[23, 55, 1]$ neither $[55, 1, 23]$ because they are isomorphic with $[1, 23, 55]$.

For the case of a trihedral group T with A such that $\#A - 1 \equiv 0 \pmod 3$, a similar description of the irreducible representations holds: three one-dimensional irreducible representation L_1, L_2, L_3 and $(\#A - 1)/3$ three-dimensional representations given by sending a generator of the group A in turn to some diagonal matrices represented by triples $[a, b, c]$. See also , [28], Chapter 3, Section 3.3 for the case when the group A is $\frac{1}{31}(1, 5, 25)$.

In a similar way as for the binary dihedral group, we define what does it mean that a polynomial (respectively a triple of polynomials) belongs to a one-dimensional irreducible representation (respectively a three-dimensional one).

Definition A.4. *Let $T := A \rtimes \mu_3$ be a trihedral group with $\#A - 1 \equiv 0 \pmod 3$.*

1. *We say that a polynomial P of $\mathbb{C}[X_1, X_2, X_3]$ belongs to a one-dimensional irreducible representation $L_i, i = 1, 2, 3$ of T if:*

$$\begin{cases} g \cdot P = P \\ \tau \cdot P = \omega^i P, \end{cases}$$

2. *We say that a triple of polynomials* (P, Q, R) *of* $\mathbb{C}[X_1, X_2, X_3]^3$ *belongs to a three-dimensional irreducible representation* $[a, b, c]$ *of* T *if:*
$$\begin{cases} Q = \tau \cdot P \\ R = \tau^2 \cdot P \\ g \cdot (P, Q, R) = (\epsilon^a P, \epsilon^b \tau \cdot P, \epsilon^c \tau \cdot P). \end{cases}$$

3. *A set of polynomials* Γ *of* $\mathbb{C}[X_1, X_2, X_3]$ *is called a* T*graph if it contains:*
   (a) *one polynomial belonging to each one-dimensional irreducible representations* L_1, L_2, L_3
   (b) *three triples of polynomials belonging to each three-dimensional irreducible representations.*

Remark A.5. Polynomial $\tau \cdot P$ is nothing else but the polynomial obtained by permuting in a cyclic way the variables i.e. X_1 becomes X_2, X_2 changes to X_3 and X_3 to X_1. The action of τ^2 on a polynomial means to apply two times this permutation, i.e. X_1 goes to X_3, X_2 goes to X_1 and X_3 to X_2. ♣

We want to find the trihedral graphs corresponding to a trihedral cluster in the $T-$Hilbert scheme of $\mathbb{A}^3$. We have to consider that a trihedral group is not a direct product of two abelian groups, but a semi-direct product. Therefore, it is natural to take three copies of an $A-$domain (i.e. of a set of polynomials in bijection with the set of all $A-$irreducible representations), but with some relations among the corresponding polynomials.

In order to give a more precise notion, let us start – following [28], Chapter 3 – with some remarks on those trihedral graphs Γ associated to a trihedral cluster supported at the origin $\{0\}$. Denote by $I(\Gamma)$ the ideal of such a $G-$graph. Then, $\mathbb{C}[X_1, X_2, X_3]/I(\Gamma)$ is the regular representation. Thus, to find Γ is the same as to find a $\mathbb{C}-$basis of $\mathbb{C}[X_1, X_2, X_3]/I(\Gamma)$.

Claim A.6. *One can always take* 1 *to be in a trihedral graph.*

Proof:

This is because the complementary set is a proper ideal and the support is zero. ■

Claim A.7. *In a trihedral graph one can take only polynomials in two of the three variables.*

Proof:

This is clear, because the trihedral group is a subgroup of $SL_3(\mathbb{C})$. We also recall that for a graph corresponding to a diagonal abelian group (see Figure 1.11), we have the “tessellation” property. In particular, we can represent in a drawing a monomial in two of the three variables by a hexagon in the plane with axes at 120° so that a $T-$graph Γ has an associated diagram $\mathcal{D}_\Gamma$. ■

Claim A.8. *A polynomial P and its "permutations" $\tau \cdot P$ and $\tau^2 \cdot P$ are in the same trihedral graph.*

Proof:

This is because of the action of μ_3, via τ. In particular, we see that if a non-constant polynomial P_1 is in a one-dimensional representation L_1, (i.e. it is $A-$invariant) then $\tau \cdot P_1$ will be in L_2 and $\tau^2 \cdot P_1$ in L_3. ∎

Claim A.9. *A triple of polynomials corresponding to a three-dimensional representation ρ of a trihedral graph Γ can be chosen to be a triple of monomials.*

Let $(p_1, \tau \cdot p_1, \tau^2 \cdot p_1), (p_2, \tau \cdot p_2, \tau^2 \cdot p_2), (p_3, \tau \cdot p_3, \tau^2 \cdot p_3)$ be three triples of polynomials corresponding to the same representation ρ in Γ. Because of the condition 3b of the definition A.4, if $(q, \tau \cdot q, \tau^2 \cdot q)$ is another triple of polynomials belonging to ρ, then q is linearly dependent on p_1, p_2, p_3, modulo $I(\Gamma)$. Now, let m_1 be a monomial that occurs in p_1. Then $(m_1, \tau \cdot m_1, \tau^2 \cdot m_1)$ is in ρ, so there exists a, a_1, a_2, a_3 in $\mathbb{C}$ such that $am_1 + a_1p_1 + a_2p_2 + a_3p_3 \equiv 0$ (mod $I(\Gamma)$). If $a_1 \neq 0$, then this means that we can replace in the $\mathbb{C}-$basis of $\mathbb{C}[X_1, X_2, X_3]/I(\Gamma)$ the triple $(p_1, \tau \cdot p_1, \tau^2 \cdot p_1)$ by $(m_1, \tau \cdot m_1, \tau^2 \cdot m_1)$. Else, if $a_1 = 0$, for any b_1, b_2, b_3 complex numbers, polynomial $b_1(p_1 - m_1) + b_2p_2 + b_3p_3$ can not be in $I(\Gamma)$. This means that the triples $(p_1 - m_1, \tau \cdot (p_1 - m_1), \tau^2 \cdot (p_1 - m_1)), (p_2, \tau \cdot p_2, \tau^2 \cdot p_2), (p_3, \tau \cdot p_3, \tau^2 \cdot p_3)$ are linearly independent, so one can take $p_1 - m_1$ instead of p_1. Repeating the procedure leads to a monomial term m of p_1 such that the triples $(m, \tau \cdot m, \tau^2 \cdot m), (p_2, \tau \cdot p_2, \tau^2 \cdot p_2), (p_3, \tau \cdot p_3, \tau^2 \cdot p_3)$ are independent modulo $I(\Gamma)$. ∎

A.2.2 Trihedral boats

The claims of the previous section have the following consequence. We consider a three-dimensional representation of $A \rtimes \mu_3$ given by $[a, b, c]$. This is the same as to give three one-dimensional representations of the group A, of associated characters χ_a, χ_b, χ_c. Let m_a, m_b, m_c be monomials associated to the $A-$representations χ_a, χ_b, χ_c. If we can choose m_a, $\tau^2 \cdot m_b$ and $\tau \cdot m_c$ independent modulo $I(\Gamma)$, then the triples $(m_a, \tau \cdot m_a, \tau^2 \cdot m_a), (\tau^2 \cdot m_b, m_b, \tau \cdot m_b), (\tau \cdot m_c, \tau^2 \cdot m_c, m_c)$ belong to the representation $[a, b, c]$ of the trihedral group $A \rtimes \mu_3-$graph. In particular, to compute a trihedral cluster, we can search for sets of monomials T_A in only two of the three variables, such that each monomial of a T_A is in one different $A-$representation.

In a certain way, a set T_A will correspond to an $A-$graph, except that it doesn't have to satisfy the condition 2 of Definition 1.21 (it is not a convex domain). Nevertheless, we can associate to a set T_A its diagram $\mathcal{D}_A$. We ask also that, for a set T_A, the "tessellation" condition is satisfied, i.e. by parallel transport, its associated diagrams realize a tessellation of the plane.

We denote by τT_A the set $\{\tau \cdot m | m \in T_A\}$ and by $\tau^2 T_A$ the set $\{\tau^2 \cdot m | m \in T_A\}$. The condition of tessellation and the bijection with the set of all $A-$irreducible representation also satisfied by τT_A and $\tau^2 T_A$. The set $T_A \cup \tau T_A \cup \tau^2 T_A$ is a natural candidate for obtaining a trihedral graph. There are two conditions to be asked.

First, for an easy description, we take in T_A only monomials, but in order to compute the syzygies and to recover the ideal $I(\Gamma)$ corresponding to a $T-$cluster, we need some relations between some triples of monomials in the same representation. This corresponds in the binary dihedral case to the relations given for each type A,B,C of possible graphs.

For a trihedral group, let f_0 be the non-constant monomial corresponding to the trivial $A-$representation in the set T_A. We suppose that there exists a monomial m in T_A such that f_0 divides m, so $m = X_i^\alpha f_0$, with X_i one of the variables occurring in f_0 and α a non-negative integer. Then, in order to define a $T-$graph, we take $X_i^\alpha(\tau^{i-1} f_0 - \tau^{i+1} f_0)$ instead of m. In particular, for f_0 it-self, instead of taking $f_0, \tau f_0, \tau^2 f_0$ take $1, f_0 + \omega\tau(f_0) + \omega^2\tau(f_0), f_0 + \omega^2\tau(f_0) + \omega\tau(f_0)$.

Second, the set $T_A \cup \tau T_A \cup \tau^2 T_A$ has cardinal $3\#A$ and it is invariant under μ_3 action. The problem is that the union before might not be disjoint, so we could have less then $\#T$ monomials in the corresponding $T-$graph. We remark also that the complementary of this set (with relations described above) is an ideal, so the union must have some "convexity" conditions, summarized in 3 and 4 of the definition bellow.

Definition A.10. *Let $T = A \rtimes \mu_3$ be a trihedral graph with $\#A - 1 \equiv 0 \pmod 3$. A set T_A is called a $T-$boat if it satisfies the following:*

1. *constant monomial 1 is not in T_A;*
2. *T_A contains only monomials in two of the three variables and all the monomials have X_1 as a factor;*
3. *if $X_1^{k_1} Y^k$ is in T_A, where Y is one of X_2, X_3 and k_1, k are two non-negative integers with $k_1 \neq 0$, then all monomials $X_1^{k_1} Y^l, l \leq k$, respectively $X_1^t, 1 \leq t \leq k_1$ are in T_A;*
4. *if $X_1^{k_1} Y^k$ and $X_1^{k_2} Y^k$ are in T_A, where Y is one of X_2, X_3 and k_1, k_2, k are non-negative integers with $0 < k_1 < k_2$, then all the monomials $X_1^{k_1+1} Y^k, \ldots, X_1^{k_2-1} Y^k$ are also in T_A;*
5. *there is a bijection $\mathrm{wt}_A : T_A \to A^\vee$ given by sending a non-constant monomial into the character of the corresponding $A-$representation; in particular there is only one $A-$invariant monomial f_0;*
6. *the diagram associated to T_A has the "tessellation" property;*
7. *$T_A, \tau T_A, \tau^2 T_A$ are three disjoint sets and their union has the following property: if m is a monomial in the union and m' divides m, then m' is also in the union;*

8. *the union $\{1\} \sqcup T_A \sqcup \tau T_A \sqcup \tau^2 T_A$ with some monomials replaced by polynomials as in the remark above, form a $T-$graph.*

Remark A.11.

1. Condition 2 is nothing else but a choice. If T_A is a trihedral boat, then τT_A is also one and $\tau^2 T_A$ also, so in order to have a not redundant list of boats, one prefers trihedral boats "based" on X_1.

2. Conditions 3, 4 and 7 have the following geometrical description. Associate to each of the $T_A, \tau T_A, \tau^2 T_A$ its diagram in the plane with axes at 120°. Then, the resulting picture is a convex domain, with no overlapping. As an example, see Figure 2.

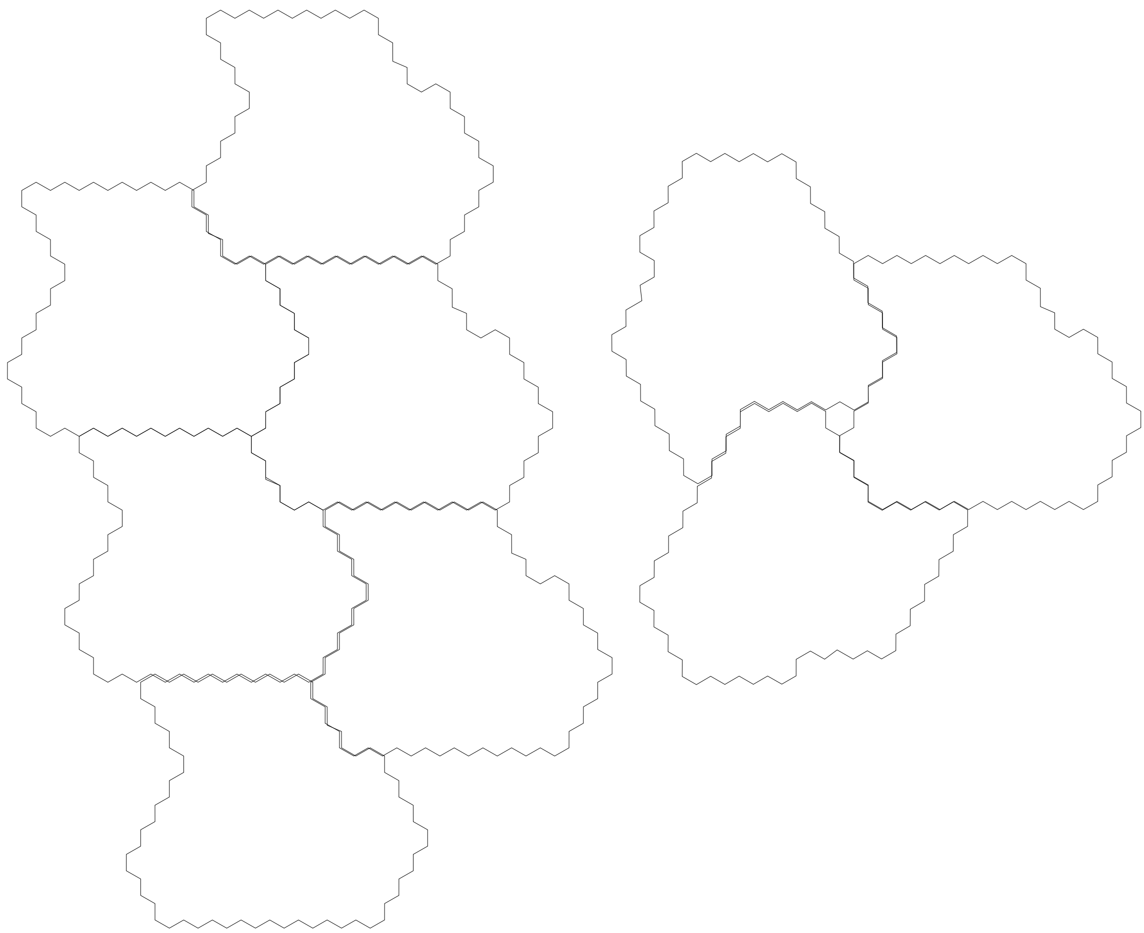

Figure A.1: Tessellation and rotation without overlapping for a trihedral boat

Here the trihedral group is $\frac{1}{79}(1, 23, 55) \rtimes \mu_3$. The little hexagon in the middle of the picture on the right is nothing else but the one corresponding to the constant monomial 1.♣

Question A.12. The previous remarks entitle us to state the following. Let T be a trihedral group of the form $A \rtimes \mu_3$, with $\#A - 1 \equiv 0 \pmod 3$. Is it then true that a $T-$graph corresponds to a $T-$cluster if and only if it can be recovered from a $T-$boat ? ♣

Example A.13. Let us see what is the result for $A = \frac{1}{79}(1, 23, 55)$. Following the definition above, we can find a list of 27 boats "based" on X_1. Of course, all the other boats will be obtained from those-ones by the action of τ. For such an example of a boat, see Figure A.2.

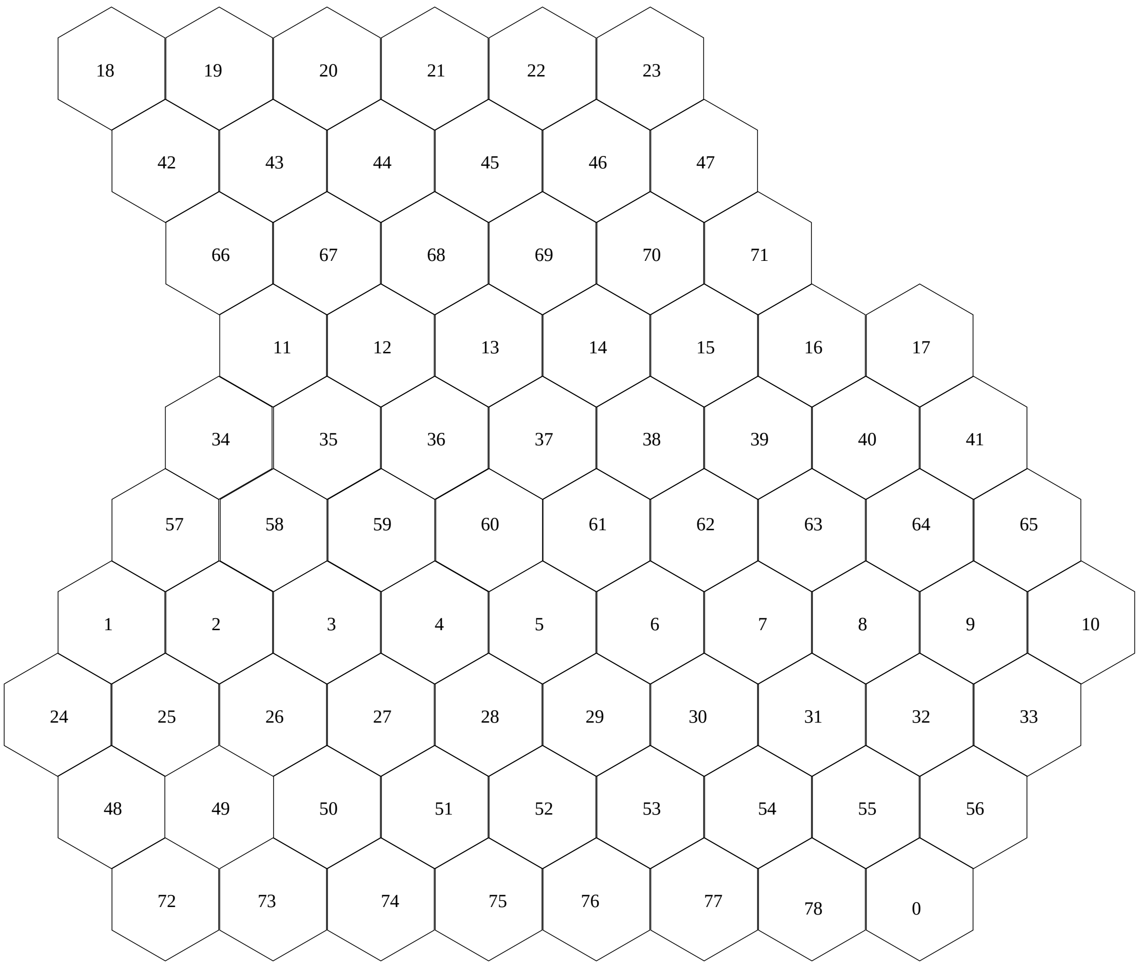

Figure A.2: A boat for the group $\frac{1}{79}(1, 23, 55) \rtimes \mu_3$

As usually, each monomial is a hexagon. The number inside each hexagon represents the weight of the associated $A-$irreducible representation. Thus, 1 is the monomial X_1, whereas 23 for example corresponds to $X_1^9 X_3^6$ because $9 \cdot 1 + 6 \cdot 55 = 23 (\text{mod } 79)$.

In this picture, the monomials on the upper left corner cf. Figure A.3 corresponds to the three-dimensional representation $[18, 19, 4]$ and give an orbifold point for the action of the crystallographic group generated by the translations and trihedral rotations.

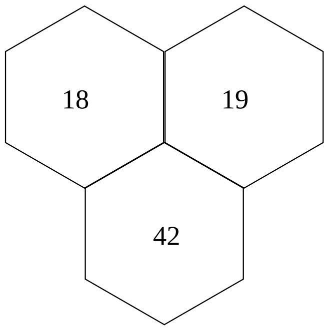

Figure A.3: Orbifold corner

The associated ideal giving a trihedral cluster at the origin is in this case the ideal with generators: $X_1^{10}X_2^3 + X_2^{10}X_3^3 + X_3^{10}X_1^3$, X_1^{11}, X_2^{11}, X_3^{11}, $X_1^{10}X_3^4$, $X_2^{10}X_1^4$, $X_3^{10}X_2^4$, $X_1^9X_3^7$, $X_2^9X_1^7$, $X_3^9X_2^7$, $X_1X_2X_3$. Computing the syzygies will give the form of a general cluster for this boat. ♣

Let us end the section by stating some open questions in analogy with the abelian case. It is natural to ask if this construction can work for a general trihedral group, i.e. with A a diagonal abelian subgroup in $SL_3(\mathbb{C})$, not necessarily of cardinal $\#A \equiv 1 \text{ mod } 3$. Another open question concerns the notion of $T-$transformation, meaning would it be possible to recover from a $T-$boat, by $T-$deformations (to be defined!) all the other boats. There are many examples that answer affirmatively to those questions, but not yet a general theory.

A.2.3 Some magma

This section includes a magma program that shows how to find trihedral boats. The idea is to find a certain set, called a Leng Polygon which contains all monomials necessary to find a trihedral boat. Once we compute this set, we make some cookery as in Claim A.2 in order to find the corresponding trihedral boat. From a trihedral boat, we can find the corresponding trihedral cluster by computing the syzygies. The programme computes the Leng Polygon. Explanations are given as a commentary in the body of the programme.

```
/////////////////////////////////////////
// Computing Leng Polygon
// GDB and MS
// 14 May 2003
// Warwick
/////////////////////////////////////////
/*
Usage
---
> Attach("leng.m");
> r := 31;
> k := 5;
> time L := LengPolygon(r,k);
Time:  6.070
> #L;
11

Input:  r,k with hcf(r,k) = 1, r | 1 + k + k^2
Thinks:  this gives A=1/r(1,k,k^2) for the trihedral group
Output:
         Leng polygon L: that is, a set of monomials such that
         L subset  x^i*y^j | i > 0, j >= 0
         L is convex
         1 in L
         some non-1 invariant in L: must do a bit more if y-power
         bigger than x power in this monomial
         for every 3-dimensional representation [a,b,c] you must
         have exactly 3 monomials in L
         EXCEPT when you are beyond the invariant monomials then
         some fiddle...
The shape of the output:
         > R := ThreeDimensionalRepresentations(31,5);
         [
           [ 1, 5, 25 ],
           [ 2, 10, 19 ],
           [ 3, 15, 13 ],
           [ 4, 20, 7 ],
           [ 6, 30, 26 ],
           [ 8, 9, 14 ],
           [ 11, 24, 27 ],
           [ 12, 29, 21 ],
           [ 16, 18, 28 ],
           [ 17, 23, 22 ]       ]
         > L := LengPolygon(31,5);
```

```
        > L;
        [
        [ [ x*y^7 ], etc.            <- sequence related to a
                                      basis of the representation
                                      [1,5,25]
        [ [ x*y^8, x^3*y^6 ], etc.]  <- ditto for the
                                      representation [2,10,19]
        ...
        [ ]
        ]
        r:=79; k:=23;

Outline
-----
We have a ring R = k[x,y].  (We do not talk about other sub-rings
of k[x,y,z].)  It has an action by a trihedral group:

               0 -> ZZ/3 -> T -> A = ZZ/r -> 0.

In particular, every monomial has an A-weight, which is in 0,...,
r-1.  This A-weight is our first piece of data.
We assume that r is congruent to 1 mod 3.  (We ignore 2 mod 3, and
0 mod 3 is different.)  Now the representation theory of such T
contains only the trivial representation and a known number of 3-
dimensional representations (as irreducibles).  These are things
like (in the case r=31)

                         [ 1, 5, 25 ]

which means
        g -> diag(1,5,25) (meaning eps_31^5, etc.)
        t -> some fixed permutation matrix.
(Here t is the image of a generator of ZZ/3, so T = < g^i*t^j >.)
In the end we want to classify T-clusters.  To do this we write
down Leng Polygons.  So this algorithm tries to write down all Leng
Polygons.

A LENG POLYGON is a subset of monomials of R=Universe(monos)
        L =  x^iy^j
where
        L is convex,
        L contains 1,
        L contains a nontrivial invariant monomial.
Now, ideally we would also ask that
        for every 3-dimensional representation rho = [a,b,c], L
```

```
         contains exactly 3 monomials whose H-weights lie in rho.
Two hitches:
         (1) (not a hitch) we won't necessarily see all three of
         a,b,c
         (2) (a hitch) sometimes we must regard two monomials as
         being the same.
We can quantify (2), and later we do in terms of 'going past ze-
ros', but in any case, for the time being, (2) forces us to work
with SEQUENCES of monomials, rather than simply monomials.  This
is a big pain in the arse.
It follows from (1) that we won't necessarily see 0,...,r-1 as
weights in our Leng polygon.  (Perhaps this will be possible in
retrospect by choosing better jigsaw pieces.)

We collect PEBBLES:
         a PEBBLE is a set [ m1, m2,...]  where mi are monomials.
Typically, a pebble has exactly one monomial.  The exceptional case
is when it has two monomials.  (there should also be triplets com-
ing from 'going past zero' near the edge.)

WARNING: for the time being, we assume only 1 or 2 monomials in
a thing.

Strategy
-----
Step 1.
      Compute all 3-dimensional representations.  Now run through
each A-invariant monomial m on the Newton polygon faces.
Step 2.
      Make the big convex hull of m and include the EXTRA BIT.
      We also figure out something of the form <L1,L2>.
Step 3.
      Recursively add an extra pebble to incomplete Leng polygons.
Step 4.
      Eradicate duplicates.
Names
---
TDR    - a sequence of irreducible 3-dimensional representations
         of T: like [ [1,5,25],...  ].
L       - a Leng polygon:  it is a sequence of sequence that are
         in bijection with the irreducible representations of T.
         Beware!  We put the trivial representation at the end.
         The order for the others is as in TDR.
Lengs   - sequence containing (partial) Leng polygons in the form
```

```
         B = < L1, L2 > where L1 is the polygon so far, and L2 is
         stuff we might add to it later
B      - see above
m      - our current A-invariant monomial

Functions
------
sort( L,C,TDR,r,k);
         takes an (empty) L and a collection C of pebbles and puts
         each pebble in the right representation.
*/
///////////////////////////////////////
// Printing
///////////////////////////////////////

intrinsic PrintLengPolygon(L::SeqEnum)
{Print the Leng polygon L}
         monos := &cat &cat L;
         R := Universe(monos);
         inds := { Exponents(m) :  m in monos };
         x_size := Maximum({ e[1] :  e in inds });
         y_size := Maximum({ e[2] :  e in inds }) + 1;
         M := Matrix(y_size,x_size, [ R | 0 :  i in [1..x_size*y_size]
]);
         for m in monos do
           E := Exponents(m);
           M[y_size - E[2],E[1]] := m;
         end for;
         print "";
         print M;
         print "";
end intrinsic;

/////////////////////////////////////
// The main function
/////////////////////////////////////

forward basicNP, convex_hull, zero_convex_hull, monomial_weight,
rep_index, ThreeDimensionalRepresentations, add_one_mono, is_shadowed,
sort, match, step3, remove, add, three_rule_ok, exact_three_rule_ok,
are_equal;

intrinsic LengPolygon(r::RngIntElt,k::RngIntElt) -> SeqEnum
```

```
{Compute the Leng polygon for the trihedral group on 1/r(1,k,k^2
mod r)}
     beach := [];
              // this is where we keep our results
     n := Floor((r-1)/3);
             // the number of 3-dimensional representations
     Lring<x,y>:=PolynomialRing(Rationals(),2);
              // a ring in which we work
     L := [ [ Parent([Lring|]) | ] :  i in [1..n+1] ];
     Lengs := [ Parent(<L,L>) | ];

// Step 1.  Just list 3-dimensional representations
     TDR := ThreeDimensionalRepresentations(r,k);

// Step 2.  Preparation step
     Nm := basicNP(r,k,Lring);
              // a sequence containing the 'primitive' invariant'
              // monomials
     for m in Nm do
              // make the convex hull with respect to m and
              // identify all representations
         C := zero_convex_hull(m);
              // makes a sequence of monomials 'under' each m
         Lcurrent := L;
         sort( Lcurrent,C,TDR,r,k);
              // Reality check:  make sure that i don't have
              // >= 4 monomials in any representation
          if not three_rule_ok(Lcurrent) then
              continue m;
          end if;
              // make a second Lng Polygon containing those
              // pebbles that we might use to include in L later:
              // we are making 'L2'.
           if &and[ #l eq 3 :  l in Lcurrent[1..n] ] then
              Append(~Lengs, <Lcurrent,L>);
              // ignore finished cases
              continue m;
           else
              Lrest := step3(Lcurrent,L,TDR,r,k,Lring,m,Nm);
              Append( Lengs, <Lcurrent,Lrest>);
           end if;
     end for;
```

```
   // Step 3.
            // go through our list Lengs of Lengs and try to top
            // up those that don't have 3 monos in every
            // (nontrivial) representation.  We are treating 'Lengs'
            // as a stack that we remove a working 'shell' from,
            // deal with it, and then put any unfinished
            // output at the bottom.
   while #Lengs ne 0 do
      B := Lengs[1];
      Remove(~Lengs,1);
            // Pick B from the top of Lengs
      if exact_three_rule_ok(B[1]) then
       Append( beach,B[1]);
            // Record it if it's already finished
      else
            // find the first (nontrivial) representation
            // that is not full
       p := 0;
       repeat
        p +:= 1;
       until #B[1][p] lt 3;
            // we removed any B that was already
            // done, so this is ok.
            // now we look for all pebbles not yet in B but that
            // have rep B[p].
            // we make a bunch of new Bs, each of which has one
            // of these new pebbles included in B[p].
       newBs := add_one_pebble(B,p,TDR,r,k);
       BBs := [ x :  x in newBs | three_rule_ok(x[1]) ];
       Lengs cat:= BBs;
      end if;
   end while;
   // Step 4.  Tidy up
            // Pretend that everything is OK by removing
            // duplicates:  nobody need know.
     final := [];
     for b in beach do
      if not &or[ are_equal(b,x) :  x in final ] then
       Append(~final,b);
      end if;
     end for;
     return final;
end intrinsic;
```

```
/////////////////////////////////////
// Auxiliary functions
/////////////////////////////////////
// x,y are both Leng Polygons.  return true iff they are the same
// thing our test is simply whether or not they involve exactly
// the same monomials.
function are_equal(x,y)
      return SequenceToSet(&cat&cat x) eq SequenceToSet(&cat&cat
y);
end function;
// we make a bunch of new Bs, each of which is B with a single ex-
tra
// monomial in B[p] -- p is just an integer telling us which en-
try
// to work on.
// We'll test each of these for the 3-rule, discard the failures
// and include the rest as partial beckys in Lengs.
forward new_people;
function add_one_pebble(B,p,TDR,r,k)
      results := [];
      Lsofar := B[1];
      Lextra := B[2];
      extra_folk := &cat Lextra;
              // we need this when computing 'shadow' below
      for mm in Lextra[p] do
              // run through possible extra pebbles
       Lnew := Lsofar;
       Lextranew := Lextra;
       to_add := new_people(mm,extra_folk);
              // computes the shadow of mm
              // must add these to Lnew and remove them
              // from Lextranew
       for aa in to_add do
        aaind := rep_index(monomial_weight(aa[1],r,k),TDR);
        add(~Lnew,aa,aaind);
        remove(~Lextranew,aa);
       end for;
       Append(~results, <Lnew,Lextranew>);
      end for;
      return results;
end function;

// say TRUE iff no representation has more than 3 pebbles in it.
function three_rule_ok(L)
```

```
        return not &or[ #l ge 4 :  l in L ];
end function;

// say TRUE iff every rep has exactly 3 pebbles in it (except mr.
trivial).
function exact_three_rule_ok(L)
        return &and[ #l eq 3 :  l in L[1..#L-1] ];
end function;

// mm a child (ie a sequence of monomials), RR a sequence
// of children (pebbles).
// return a seq ctg any young person in RR that is <= mm.
// (recall:  <= means ...)
forward child_le;
function new_people(mm,RR)
      result := [ Parent(mm) | ];
      for rr in RR do
       if child_le(rr,mm) then
        Append(~result,rr);
       end if;
      end for;
// add some extra pure x^i powers to match the largest power
// of y in mm
// NOTE: next lines are OK because pure x^i monomials appear
// as [x^i] unipebbles.
      _<x,y> := Universe(mm);
      poss_xi := [ m :  m in RR | Degree(m[1],y) eq 0 ];
      max_y := Maximum( [ Degree( m , y ) :  m in mm ] );
      for xx in poss_xi do
       if Degree(xx[1],x) le max_y then
        Append(~result,xx);
       end if;
      end for;
      return result;
end function;

function child_le(aa,bb)
      for a in aa do
       for b in bb do
        a1,a2 := Explode(Exponents(a));
        b1,b2 := Explode(Exponents(b));
        if a1 le b1 and a2 le b2 then
           return true;
        end if;
```

```
     end for;
    end for;
    return false;
end function;

function step2(Lcurrent,L,TDR,r,k,Lring,m,Nm)
    x := Lring.1;
    y := Lring.2;
    Lrest := L;
    current_monos := &cat &cat Lcurrent;
    other_monos := [Lring| ];
    for a in [1..r-1] do
     for b in [0..r-1] do
      if x^a*y^b in current_monos then
        continue b;
      elif is_shadowed(a,b,Nm,m) then
        continue a;
      end if;
      Append(~other_monos,x^a*y^b);
     end for;
    end for;
    sort(~Lrest,other_monos,TDR,r,k);
    match(~Lrest,m,r,k,TDR);
    return Lrest;
end function;

// L is a sequence of sequences of monomials in representations
// m is a mono
// if we find a monomial sequence [m1] in L with
procedure match( L,m,r,k,TDR)
    i,j := Explode(Exponents(m));
    k1 := 1;
    Lring<x,y> := Parent(m);
    all_monos := &cat &cat L;
    done := false;
    repeat
     m_right := x^(i+k1)*y^j;
     m_up := x^i*y^(j+k1);
     if m_right not in all_monos then
// remove all further m_up's
      while x^i*y^(j+k1) in all_monos do
        remove(~L,[x^i*y^(j+k1)]);
        k1 +:= 1;
      end while;
```

```
         done := true;
       elif m_up notin all_monos then
// remove all further m_right's
         while x^(i+k1)*y^j in all_monos do
           remove(~L,[x^(i+k1)*y^j]);
           k1 +:= 1;
         end while;
         done := true;
       else
         ri_r := rep_index(monomial_weight(m_right,r,k),TDR);
         ri_u := rep_index(monomial_weight(m_up,r,k),TDR);
         if ri_r eq ri_u then
// include [m_right,m_up] as a new rep and remove [m_r],[m_u]
           remove(~L,[m_right]);
           remove(~L,[m_up]);
           add(~L,[m_right,m_up],ri_r);
         else
// remove [m_r],[m_u] and everybody else upper and righter.
           done_r := false;
           done_u := false;
           repeat
             if not done_r and x^(i+k1)*y^j in all_monos then
               remove(~L,[x^(i+k1)*y^j]);
             else
               done_r := true;
             end if;
             if not done_u and x^i*y^(j+k1) in all_monos then
               remove(~L,[x^i*y^(j+k1)]);
             else
               done_u := true;
             end if;
             k1 +:= 1;
           until done_u and done_r;
           done := true;
       end if;
      end if;
      k1 +:= 1;
      until done;
end procedure;

// L is a sequence of seqs of pebbles in reps
// R is a sequence [m], that is, a pebble
// remove R from L if possible -- don't worry if you don't find
```

```
it.
procedure remove(~L,R)
      for i in [1..#L] do
       l := L[i];
       for x in l do
        if x[1] in R then
          Remove(~l,Index(l,x));
          L[i] := l;
          return;
        end if;
       end for;
      end for;
end procedure;

// L is a sequence of seqs of monos in reps
// R is a sequence [m1,m2]
// add R to L in the right representation, position p.
procedure add(~L,R,p)
      l := L[p];
      Append(~l,R);
      L[p] := l;
end procedure;

// L is a sequence of seqs of monomials in representations
// M is a seq of monomials
// put the guys in M into the right place in L
procedure sort(~L,M,TDR,r,k)
      for mono in M do
        wt := monomial_weight(mono,r,k);
        place := rep_index(wt,TDR);
        if wt ne 0 then
          Include(~L[place],[mono]);
        else
          Include(~L[#TDR+1],[mono]);
        end if;
      end for;
end procedure;

function is_shadowed(a,b,N,m)
      for n in N do
        k,l := Explode(Exponents(n));
        if n eq m then
          if a gt k and b gt l then
            return true;
```

```
            end if;
         elif a ge k and b ge l then
            return true;
         end if;
      end for;
      return false;
end function;

// the index of the representation from among reps T containing
weight w
function rep_index(w,T)
      for i in [1..#T] do
         if w in T[i] then
              return i;
         end if;
      end for;
      return 0;
end function;

// the weight of a monomial m w.r.t.  action 1/r(1,k,k^2(bar))
function monomial_weight(m,r,k)
      i,j := Explode(Exponents(m));
      return (i + j*k) mod r;
end function;

// compute the convex hull of a monomial m
function convex_hull(m)
      Lring<x,y> := Parent(m);
      i,j := Explode(Exponents(m));
      return [ x^i0*y^j0 :  i0 in [1..i], j0 in [0..j] ];
end function;

function zero_convex_hull(m)
      Lring<x,y> := Parent(m);
      i,j := Explode(Exponents(m));
      return [ x^i0*y^j0 :  i0 in [1..i], j0 in [0..j] ] cat [x^k
:  k in [i+1..j]];
end function;

//Newton polygon in xOy plane
function basicNP(r,k,L)
      x := L.1;
      y := L.2;
      Nm:=[x^(r-k)*y, x^r];
```

```
      while Degree(Nm[1],x) gt 0 do
       h1:=Nm[1]; h2:=Nm[2]; a:=Ceiling(Degree(h2,x)/Degree(h1,x));
       Nm:=[L!(h1^a/h2)] cat Nm;
      end while;
      Nm:=Reverse(Nm);
      return Prune(Nm);
end function;

//representations
// R is 3-dimensional representations
// B is auxiliary and irrelevant.
function ThreeDimensionalRepresentations(r,k)
     B:=[0:  i in [1..r-1]]; R:=[];
     for i in [1..r-1] do
      if B[i] eq 0 then
       R:=R cat [[i,(i*k) mod r,(i*k^2) mod r]];
       B[i]:=1; B[(i*k) mod r]:=1; B[(i*k^2) mod r]:=1;
      end if;
     end for;
     return R;
end function;
```

Appendix B

List of notations

$G_n = \mu_{2^n-1}$	the cyclic group of order $2^n - 1$
ϵ	a fixed primitive root of unity of order $2^n - 1$
g_n	the diagonal matrix $\mathrm{diag}(\epsilon, \epsilon^2, \epsilon^{2^2}, \ldots, \epsilon^{2^{n-1}})$
H_n	the group μ_{2^n-1} as subgroup of $SL_n(\mathbb{C})$ with generator g_n
h_n	the vector $\frac{1}{2^n-1}(1, 2, 2^2, \ldots, 2^{n-1})$
$\{e_1, \ldots, e_n\}$	canonical basis of $\mathbb{R}^n$ ($\mathbb{Z}^n$)
$2^i \star h_n$	the vector $\frac{1}{2^n-1}(2^i, 2^{i+1}, \ldots, 2^{n-1}, \underbrace{1}_{(n-i+1)^{\text{th}} \text{ position}}, 2, 2^2, \ldots, 2^{i-1})$
ω_X	the canonical sheaf of the variety X
K_X	canonical divisor of the variety X
N	a lattice in Chapter 1, a functor defined in Lemma 2.86
$N^\vee$	the dual of a lattice (1.1.1)
M	the semigroup of all monomials in n variables (1.2.1)
Δ	a fan
σ, τ	cones in a fan
σ_0	the cone generated in a lattice $N \supset \mathbb{Z}^n$ by the canonical basis $\{e_1, \ldots, e_n\}$
ρ	a ray in a cone or an [irreducible] representation
χ	a character
Δ_n	the $n-$dimensional simplex
Γ	a $G-$graph, for G finite group
$\mathcal{D}(\Gamma)$	planar diagramme associated to a $G-$graph, for $G \subset SL_3(\mathbb{C})$
$\mathrm{Graph}(G)$	the set of all $G-$graphs
$w(m)$	weight of a Laurent monomial
$\mathrm{wt}_\Gamma(m)$	unique monomial of Γ with the same associated character as m
$r_\Gamma(m', m)$	with $m' \notin \Gamma$, $m \in \Gamma$ (Definition 1.25)
$m/\mathrm{wt}_\Gamma(m)$	ratio of m with respect to Γ
$D_m(\Gamma)$	deformation of Γ along m
$\mathrm{mp}_S(X)$	maximal power of the variable X in the set S (Definition 1.28)
$\mathrm{pv}(S)$	power vector of the set S (Definition 1.28)

$\sigma(\Gamma)$	cone associated to a $G-$graph Γ (Definition 1.29)
$\mathrm{Fan}(G)$	the set of all the cones $\sigma(\Gamma)$ when Γ runs over $\mathrm{Graph}(G)$
$S(\Gamma)$	semigroup associated to a $G-$graph Γ (idem)
$V(\Gamma)$	$\mathrm{Spec}\mathbb{C}[S(\Gamma)]$ (idem)
$I(\Gamma)$	ideal generated by the monomials that are not in Γ
$q(t), r(t)$	indices in Lemma 1.43
F^a	sheafification of a functor
$\mathcal{P}_X$	see Proposition 2.83
$\mathcal{X}$	smooth Deligne-Mumford stack
$\mathcal{T}$	a topology on a site
h_W	the contravariant functor $\mathrm{Hom}(\cdot, W)$, for W object in a category
$\mathcal{C}$	a category (in most of the cases a site)
$\mathcal{F}$	fibered category
$\mathrm{Ram}(f) = R(f)$	ramification locus of a map f
$\mathcal{U} = \{a_i : U_i \to U\}$	covering of a variety U
p_i	projection on the i^{th} factor of a [fiber] product
f^*	pullback of f
p	a prime number, a monomial, an index or the first projection
$a \pmod b$	the remainder of a modulo b, an integer in the set $\{0, \ldots, b-1\}$
$\mathcal{A}_s, \mathcal{S}_s, \mathcal{R}_s, \mathcal{C}_s$ etc.	sets of cones (3.1.2)
#	cardinal of a set
Φ_k	Fourier-Mukai functor, with kernel K
κ	algebraically closed field of prime characteristic
$\mathcal{O}_x$	skyscraper sheaf on x
F_U	$2-$functor on groupoids associated to the scheme U (2.4.2.1)

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DISCIPLINE: mathématiques

RÉSUMÉ:

Le premier chapitre montre par des méthodes toriques ($G-$graphes) que pour tout entier positif n, le quotient de l'espace affine à n dimensions par le groupe cyclique G_n d'ordre $2^n - 1$ admet le G_n-schema de Hilbert comme résolution lisse crepante. Le deuxième chapitre contient des résultats sur les champs algébriques (construction du champ algébrique lisse associé à une log-paire). Le troisième chapitre montre l'équivalence entre la catégorie dérivée bornée des faisceaux cohérents G_n-équivariants sur l'espace affine et celle des faisceaux cohérents sur la résolution G_n-Hilb. Chapitre 4 donne une réalisation géométrique de la conjecture de Broué via la correspondance de McKay. L'annexe contient des résultats sur les groupes trihédraux, y compris un programme magma.

ABSTRACT:

The first chapter shows by toric methods ($G-$graphs) that for any positive integer n, the quotient of the affine $n-$dimensional space by the cyclic group G_n of order $2^n - 1$ has the G_n-Hilbert scheme as smooth crepant resolution. The second chapter contains results on algebraic stacks (construction of a smooth algebraic stack associated to a log-pair). The third chapter shows the equivalence of the bounded derived category of G_n-equivariant coherent sheaves on the affine space with that of coherent sheaves on the resolution G_n-Hilb. Chpater 4 gives a geometric equivalent of Broué's conjecture via the McKay correspondence. The Annexe contains results on trihedral groups, including a magma programme.

MOTS-CLÉ:

groupe cyclique, crépance, variété torique, résolution de singularité, G-graphe, correspondance de McKay, champs algébriques, champs algébriques lisses associés à une log-paire, sheafification, équivalence, catégorie dérivée, conjecture de Broué, groupe trihédraux, magma

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